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ORIGNAL ARTICLE

ON (m, n)−ABSORBING IDEALS IN AN ALMOST DISTRIBUTIVE LATTICE

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Abstract

This paper explores the concept of (m, n)−absorbing ideals within an Almost Distributive Lattice (ADL). It also introduces and examines the notion of weakly (m, n)−absorbing ideals, a more generalized form of (m,n)- absorbing ideals. The primary focus is on establishing the relationship between (m, n)-absorbing ideals (and weakly (m, n)-absorbing ideals) and their counterparts, (m, n)-absorbing prime ideals (and weakly (m, n)-absorbing prime ideals), in an ADL. Additionally, the paper investigates the properties of homomorphic images and inverse images of (m, n)-absorbing ideals, demonstrating that these images retain the structure of (m, n) -absorbing ideals.

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1. Introduction

Prime ideals play a vital role in analyz ing the structure and characteristics of rings and lattices, initially introduced by Dubey (2012) and the concept has been expanded and generalized in (Anderson and Smith, 2003; Dubey, 2012; Mahdou *et al*., 2020) works. Koc et al. (2023) introduced 1-absorbing prime ideals, building on the idea of weakly 1-absorbing prime ideals from Yassine *et al.* (2021). Research on prime ideals in lattices aims to extend the concept of prime numbers, understand localization, establish representation theorems, and delve into order theory. This work enhances our grasp of mathematical structures in a more refined way. Prime ideals are also fundamental for understanding the ordering relationships in partially ordered sets (posets), aiding in the development of a robust theory of ordered structures. Let $m, n \in Z^+$ with $m > n$ and and I be a proper ideal in L. JALAL ABADI and Moghimi, (2017) introduced the concepts of an (m, n)-absorbing ideals F in commutative rings R if for all $a_1, \ldots, a_s \in R$ and $a_1 \ldots a_s \in F$ with $m \leq s$, then there are n of the a'_is whose product is in F. Additionally, the notion of weakly (m, n)-prime ideals in commutative rings was introduced in Anderson and Smith (2003), highlighting their importance in the structural theory of distributive lattices, particularly within Boolean algebras.

Swamy and Rao (1981) have introduced the concept of Almost Distributive Lattices (ADLs). According to Swamy and Rao (1981), a proper ideal F of an ADL L is prime if for all $a, b \in L$ with $a \wedge b \in F$, then $a \in F$ or $b \in F$. This concept was generalized by Natnael Teshale Amare (2022) with weakly prime ideals in L. According to Teshale Amare (2022), a proper ideal F in L is a $2-$absorbing ideal (or weakly $2-$absorbing ideal) of L if for all $a, b, c \in L$ with $a \wedge b \wedge c \in F (0 \neq a \wedge b \wedge c \in F)$, then $a \wedge b \in F$ or $a \wedge c \in F$ or $b \wedge c \in F$. In general, a proper ideal F in L is an $n-$absorbing ideal in L if for all $a_1, ..., a_n \in R$ and $a_1 \wedge ... \wedge a_n \in F$, then there are $n+1$ of the $a_i^{'}$s whose meet is in F. In this paper, we introduce the concepts of $(m, n)-$absorbing ideals and weakly $(m, n)-$absorbing ideals in an ADL L, for all $n \leq m$, which generalize $(m, n)-$absorbing prime ideals and weakly $(m, n)-$absorbing prime ideals, respectively. We establish the relationships between $(m, n)-$absorbing ideals, $(m, n)-$absorbing prime ideals and $(m, m-n)-$absorbing ideals. In addition, we note that every $(m, n)-$absorbing prime ideal is an $(m+1, n+1)-$absorbing ideal. Counterexamples are provided to illustrate that the converse is not necessarily true. Moreover, we study the relation between the pairwise co-maximal ideal and $(m, n)-$absorbing ideal and also, between minimal $(m, n)-$absorbing ideal and $(m, n)-$absorbing ideal. We observe that the intersection of a family of $(m, n)-$absorbing ideals remains an $(m, n)-$absorbing ideal. Finally, we demonstrate that homomorphic images and inverse images of $(m, n)-$absorbing ideals are also $(m, n)-$absorbing ideals. We also show that every $(m, n)-$absorbing ideal is a weakly $(m, n)-$absorbing ideal, but the converse is not always true. Finally, we establish that the ideals of a pair are weakly $(m, n)-$absorbing ideals if their direct product is a weakly $(m, n)-$absorbing ideal, yet counterexamples illustrate that weakly $(m, n)-$absorbing ideals can exist whose direct product is not necessarily a weakly $(m, n)-$absorbing ideal.

2 Preliminaries

In this section, we recall certain definitions and results concerning on an ADL, prime ideals and $(m, n)-$prime ideals which will be used in the sequel.

Definition 2.1. *Swamy and Rao (1981) An algebra $R = (R, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is referred to as an ADL if it meets the subsequent conditions for all r, s and t in R.*

1. $0 \wedge r = 0$
2. $r \vee 0 = r$
3. $r \wedge (s \vee t) = (r \wedge s) \vee (r \wedge t)$
4. $r \vee (s \wedge t) = (r \vee s) \wedge (r \vee t)$
5. $(r \vee s) \wedge t = (r \wedge t) \vee (s \wedge t)$
6. $(r \vee s) \wedge s = s$.

Every distributive lattice with a lower bound is categorized as an ADL.

Example 2.2. *Swamy and Rao (1981) For any nonempty set A, it's possible to transform it into an ADL that doesn't constitute a lattice by selecting any element 0 from A and fixing an arbitrary element $u_0 \in R$. For every $u, v \in R$, define $\wedge$ and $\vee$ on R as follows:*

$$u \wedge v = \begin{cases} v & \text{if } u \neq u_0 \\ u_0 & \text{if } u = u_0 \end{cases} \quad \text{and} \quad u \vee v = \begin{cases} u & \text{if } u \neq u_0 \\ v & \text{if } u = u_0 \end{cases}$$

*Then $(A, \wedge, \vee, u_0)$ is an ADL (called the **discrete ADL**) with u_0 as its zero element.*

Definition 2.3. *Swamy and Rao (1981) Consider $R = (R, \wedge, \vee, 0)$ be an ADL. For any r and $s \in R$, establish $r \leq s$ if $r = r \wedge s$ (which is equivalent to $r \vee s = s$). Then $\leq$ is a partial order on R with respect to which 0 is the smallest element in R.*

Theorem 2.4. *Swamy and Rao (1981) The following conditions are valid for any r, s and t in an ADL R.*

(1) $r \wedge 0 = 0 = 0 \wedge r$ and $r \vee 0 = r = 0 \vee r$

(2) $r \wedge r = r = r \vee r$

(3) $r \wedge s \leq s \leq s \vee r$

(4) $r \wedge s = r$ iff $r \vee s = s$

(5) $r \wedge s = s$ iff $r \vee s = r$

(6) $(r \wedge s) \wedge t = r \wedge (s \wedge t)$ (in other words, $\wedge$ is associative)

(7) $r \vee (s \vee r) = r \vee s$

(8) $r \leq s \Rightarrow r \wedge s = r = s \wedge r$ (iff $r \vee s = s = s \vee r$)

(9) $(r \wedge s) \wedge t = (s \wedge r) \wedge t$

(10) $(r \vee s) \wedge t = (s \vee r) \wedge t$

(11) $r \wedge s = s \wedge r$ iff $r \vee s = s \vee r$

(12) $r \wedge s = \inf\{r, s\}$ iff $r \wedge s = s \wedge r$ iff $r \vee s = \sup\{r, s\}$.

Definition 2.5. *Swamy and Rao (1981) Let R and G be ADLs and form the set $R \times G$ by $R \times G = \{(r, g) : r \in R$ and $g \in G\}$. Define $\wedge$ and $\vee$ in $R \times G$ by, $(r_1, g_1) \wedge (r_2, g_2) = (r_1 \wedge r_2, g_1 \wedge g_2)$ and $(r_1, g_1) \vee (r_2, g_2) = (r_1 \vee r_2, g_1 \vee g_2)$, for all $(r_1, g_1), (r_2, g_2) \in R \times G$. Then $(R \times G, \wedge, \vee, 0)$ is an ADL under the pointwise operations and $0 = (0, 0)$ is the zero element in $R \times G$.*

Definition 2.6. *Swamy and Rao (1981) Let R and G be ADLs. A mapping $f : R \to G$ is called a homomorphism if the following are satisfied, for any $r, s \in R$.*
(1). $f(r \wedge s) = f(r) \wedge f(s)$
(2). $f(r \vee s) = f(r) \vee f(s)$
(3). $f(0) = 0$.

Definition 2.7. *Swamy and Rao (1981) A non-empty subset, denoted as I in an ADL R is termed an ideal in R if it satisfies the conditions: if u and v belong to I, then $u \vee v$ is also in F, and for every element r in R, the $u \wedge r$ is in F.*

Definition 2.8. *Swamy and Rao (1981) A proper ideal I in R is a prime ideal if for any u and v belongs R, $u \wedge v$ belongs F, then either u belongs F or v belongs F.*

Theorem 2.9. *Stone (1938) Let I be an ideal in R. Let F be a non-empty subset in R such that $r \wedge s \in F$, for all r and $s \in F$. Assume $I \cap F$ is empty set. Then there exists a prime ideal P in R containing I and $P \cap F$ is empty set.*

Theorem 2.10. *Teshale Amare (2022) Let P be an ideal in R. Then P a weakly prime ideal in R only if P is a prime ideal in R.*

Definition 2.11. *Khashan and Celikel (2024a) Let R be a ring and m, n be positive integers. A proper ideal I of R is called a $(m, n)-$prime in R if for $a, b \in R, a^m b \in I$ implies either $a^n \in I$ or $b \in I$..*

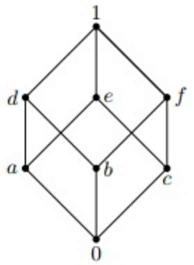
Definition 2.12. *Khashan and Celikel (2024b) Let R be a ring and m, n be positive integers. A proper ideal I of R is called weakly $(m, n)-$prime in R if for $a, b \in R, 0 \neq a^m b \in I$ implies either $a^n \in I$ or $b \in I$..*

3 $(m, n)-$Absorbing Ideals

In this section, we define and characterize the concept of $(m, n)-$absorbing ideals in an ADL L and their properties. In particular, we study on the direct product of $(m, n)-$absorbing ideals and their homomorphic images.

Definition 3.1. *Let $m, n \in Z^+$ with $m > n$. A proper ideal F in L is an $(m, n)-$absorbing ideal in L if for all $f_1, f_2, ..., f_m \in L$ such that $\bigwedge_{i=1}^{m} f_i \in F$, then there are n of $f_i's$ whose meet is in F.*

Example 3.2. *Let $L = \{0, u, v\}$ be a discrete ADL with 0 as its zero element defined in Example 2.2 and $M = \{0, a, b, c, d, e, f, 1\}$ be a lattice whose Hasse diagram is given below:*



Put $G = \{(0,0),(0,b),(0,a),(0,d)\}$. Clearly G is an ideal in $L \times M$. For any element $(u,d),(v,e),(0,f) \in L \times M$, $(u,d)\wedge(v,e)\wedge(0,f) \in G$ implies $(u,d)\wedge(0,f) \in G$. Thus G is a $(3,2)-$absorbing ideal in $L \times M$.

Let us recall that, $\langle r] = \{r \wedge s : s \in L\}$.

Lemma 3.3. *Let $f_i, g_i \in L$, for all $1 \le i \le m$. Then the following hold.*

(1). $\bigcap_{i=1}^{m} \langle f_i] = \langle \bigwedge_{i=1}^{m} \langle f_i]$

(2). $\langle \bigwedge_{i=1}^{m} \langle f_i] \cap \langle \bigwedge_{i=1}^{m} \langle g_i] = \langle \bigwedge_{i=1}^{m} (f_i \wedge g_i)] = \langle \bigwedge_{i=1}^{m} (g_i \wedge f_i)]$

(3). $\langle \bigwedge_{i=1}^{m} \langle f_i] \vee \langle \bigwedge_{i=1}^{m} \langle g_i] = \langle \bigwedge_{i=1}^{m} (f_i \vee g_i)] = \langle \bigwedge_{i=1}^{m} (g_i \vee f_i)]$.

Theorem 3.4. *Let F be a proper ideal in L. Then the following are equivalent.*

(1). F is an $(m,n)-$absorbing ideal

(2). For any ideals $F_1,...,F_m \in L$ and $\bigcap_{i=1}^{m} F_i \subseteq F$, then there are the n of $F_i^{'}$s whose intersection is a subset of F

(3). For any ideals $F_1,...,F_m \in L$ and $F = \bigcap_{i=1}^{m} F_i$, then there are the n of $F_i^{'}$s whose intersection is a subset of F.

Proof. (1) $\Leftrightarrow$ (2) : Suppose F is an $(m,n)-$absorbing ideal. Let $F_1,...,F_m$ be ideals in L such that $\bigcap_{i=1}^{m} F_i \subseteq F$. Assume there are n of $F_i^{'}$s whose intersection is not a subset of F; that is, $\bigcap_{i=1}^{n} F_i \nsubseteq F$, $\bigcap_{i=2}^{n+1} F_i \nsubseteq F$ and so on. Choose $f_i \in L$, for each $1 \le i \le n$, $\bigwedge_{i=1}^{n} f_i \in \bigcap_{i=1}^{n} F_i$, $\bigwedge_{i=1}^{n} f_i \notin F$, $\bigwedge_{i=2}^{n+1} f_i \in \bigcap_{i=2}^{n+1} F_i$, $\bigwedge_{i=2}^{n+1} f_i \notin F$ and so on. Consequently, $\bigwedge_{i=1}^{m} f_i \in \bigcap_{i=1}^{m} F_i$ and $\bigwedge_{i=1}^{m} F_i \notin F$. Thus, $\bigcap_{i=1}^{m} F_i \nsubseteq F$, gives a contradiction. Hence the result. On the other hand, let $f_1,...,f_m \in L$ such that $\bigwedge_{i=1}^{m} f_i \in F$. So, $\langle \bigwedge_{i=1}^{m} f_i] \subseteq F$. By assumption, we get that there are n of the $\langle f_i]'$s whose intersection is a subset of F and hence there are n of the f_i's whose meet is in F, since $f_i \in \langle f_i]$ and by lemma 3.3 (1). Thus, F is an $(m,n)-$absorbing ideal in L.

(2) $\Leftrightarrow$ (3) and (3) $\Leftrightarrow$ (1) are clear. □

Theorem 3.5. *Let F be a proper ideal in L. Then the following assertion hold.*

(1). F is a $2-$absorbing ideal iff F is a $(3,2)-$absorbing ideal

(2). If F is an (m,n)-prime ideal, then F is an $(m+1,n+1)-$absorbing ideal

(3). If F is an $(m,n)-$absorbing prime ideal, then F is an $(m,n)-$absorbing ideal

(4). If F is an $(m,n)-$absorbing ideal, then F is an $(m^*,n^*)-$absorbing ideal, for all $m^* \ge m$ and $n^* \ge n$

(5). F is an $(m,n)-$absorbing ideal iff F is an $(m,m-n)-$absorbing ideal.

Proof. (1). For $m = 3$ and $2 = 1$, it is clear.

(2). Suppose F is an (m,n)-prime ideal. Let $f_1,f_2,...,f_{m+1} \in L$ with $\bigwedge_{i=1}^{m+1} f_i \in F$. Then $\bigwedge_{i=1}^{m+1} f_i = \bigwedge_{i=1}^{m} f_i \wedge f_{m+1} \in F$ and hence $\bigwedge_{i=1}^{n} f_i \in F$ or $f_{m+1} \in F$, since F is $(m,n)-$prime. Thus, Q is an $(m+1,n+1)-$absorbing ideal.

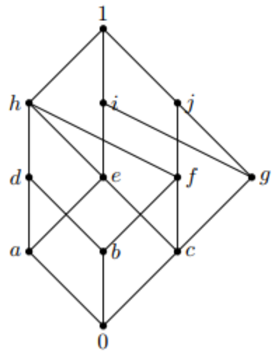
(3). Suppose F is an $(m,n)-$absorbing prime ideal. Let $f_1,f_2,...,f_m \in L$ such that $\bigwedge_{i=1}^{m} f_i \in F$. Then either $\bigwedge_{i=1}^{n} f_i \in F$ or $\bigwedge_{i=n+1}^{m} f_i \in F$. Hence the result.

(4). Suppose F is an $(m,n)-$absorbing ideal in L and $m^* \ge m$ and $n^* \ge n$, for all

$m, n, m^*, n^* \in Z^+$. Let $f_1, f_2, ..., f_m \in L$ with $\bigwedge_{i=1}^{m^*} f_i \in F$. We prove there are n^* of f_i's whose meet is in F.
Case 1: Let $m^* = m$. It is clear.
Case 2: Let $m^* > m$. If $n^* = n$, then there are n of f_i's whose meet is in F. Suppose $n^* > n$. By assumption, we get that $\bigwedge_{i=1}^{n} f_i \in F$ or $\bigwedge_{i=2}^{n+1} f_i \in F$ or $\bigwedge_{i=3}^{n+2} f_i \in F$ and so on. It follows that $\bigwedge_{i=1}^{n^*} f_i = \bigwedge_{i=1}^{n} f_i \wedge f_{n^*} \in F$, $\bigwedge_{i=2}^{n^*+1} f_i = \bigwedge_{i=2}^{n+1} f_i \wedge f_{n^*+1} \in F$ and so on (since F is an ideal and by hypothesis). Thus there are n^* of f_i's whose meet is in F. Therefore, F is an $(m^*, n^*)-$absorbing ideal.
(5). Suppose H is an $(m,n)-$absorbing ideal. Let $f_1, f_2, ..., f_m \in L$ with $\bigwedge_{i=1}^{m} f_i \in F$.
Case 1. Let $m - n > n$. It is clear, by assumption.
Case 2. Let $n > m - n$. Then $\bigwedge_{i=1}^{n} f_i = \bigwedge_{i=1}^{m-n} f_i \wedge \bigwedge_{i=m-n+1}^{n} f_i$. By assumption to get that $\bigwedge_{i=1}^{n} f_i \in F$ or $\bigwedge_{i=2}^{n+1} f_i \in F$ or $\bigwedge_{i=3}^{n+2} f_i \in F$ and so on. Thus, $\bigwedge_{i=1}^{m-n} f_i \wedge \bigwedge_{i=m-n+1}^{n} f_i \in F$. Since F is an ideal, we have $\bigwedge_{i=1}^{m-n} f_i \in F$ or $\bigwedge_{i=m-n+1}^{n} f_i \in F$. Thus, F is an $(m, m-n)-$absorbing ideal. The converse is clear. □

The converse of the above results (2-4) are not true; consider the following example.

Example 3.6. *Let $C = \{0, u, v\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $D = \{0, a, b, c, d, e, f, g, h, i, j, 1\}$ be a lattice whose Hasse diagram is given below:*



Consider $C \times D = \{(t, s) : t \in C$ and $s \in D\}$. Then $(C \times D, \wedge, \vee, 0)$ is an ADL (note that $C \times D$ is not a lattice) under the point-wise operations $\wedge$ and $\vee$ on $C \times D$ and $0 = (0, 0)$, the zero element in $C \times D$.
(2). Put $A = \{(0, 0), (0, a)\}$. Note that A is an ideal in $C \times D$. Let $(u, d), (v, e), (u, f), (0, g), (v, h) \in C \times D$ such that $(u, d) \wedge (v, e) \wedge (u, f) \wedge (0, g) \wedge (v, h) \in A$ implies $(u, d) \wedge (0, g) \wedge (v, h) = (0, 0) \in A$. Thus A is a $(5, 3)-$absorbing ideal in $C \times D$. But A is not a $(4, 2)$-prime ideal, since $(u, d) \wedge (v, e) \wedge (u, f) \wedge (0, g) \wedge (v, h) \in A$ implies $(u, d) \wedge (v, e) = (u, a) \notin A$, $(u, f) \wedge (0, g) = (0, c) \notin A$ and $(v, h) \notin A$.
(3). Clearly A is a $(4, 2)-$absorbing ideal, A is defined in above (2). While A is not a $(4, 2)-$absorbing prime ideal in $C \times D$, since $(u, d) \wedge (v, e) \wedge (u, f) \wedge (0, g) \in A$ implies $(u, d) \wedge (v, e) = (u, a) \notin A$ and $(u, f) \wedge (0, g) = (0, c) \notin A$.
(4). Let $B = \{(0, 0)\}$. Clearly B is a $(5, 3)-$absorbing ideal but not a $(4, 2)-$absorbing ideal. This is demonstrated by considering any elements $(0, d), (u, e), (v, f), (u, h) \in C \times D$ with $(0, d) \wedge (u, e) \wedge (v, f) \wedge (u, h) \in B$. Then $(0, d) \wedge (u, e) = (0, a) \notin H$, $(0, d) \wedge (v, f) = (0, b) \notin B$, $(0, d) \wedge (u, h) = (0, d) \notin B$, $(u, e) \wedge (v, f) = (u, c) \notin B$, $(u, e) \wedge (u, h) = (u, e) \notin B$ and $(v, f) \wedge (u, h) = (u, c) \notin B$.

(1). An $(n+1, n)-$absorbing ideal is just $n-$absorbing ideal and in particular, a $(2, 1)-$absorbing ideal is just a prime ideal
(2). Every $(m, n)-$absorbing ideal is a $(m-1)-$absorbing ideal

(3). The intersection of $n-$prime ideals is $n-$absorbing ideals
(4). If F_j is an n_j-absorbing ideal, for each $1 \leq j \leq m$, then $\bigcap_{j=1}^{m} F_j$ is a $n-$absorbing ideal, for each $n = \bigvee_{i=1}^{m} n_i$. In particular, if $F_1, ..., F_n$ are prime ideals in L, then $\bigcap_{i=1}^{n} F_i$ is an $n-$absorbing ideal.

Let F be an ideal in L. We note that, $F \cap \langle r] = \{s \in L : r \wedge s \in F\}$.

Lemma 3.7. *Let F be an $(m,n)-$absorbing ideal in L. Then $F \cap \langle r]$ is an $(m,n)-$absorbing ideal in L containing F.*

Proof. Let $f_1, ..., f_m \in L$ with $\bigwedge_{i=1}^{m} f_i \in F \cap \langle r]$. Then $r \wedge \bigwedge_{i=1}^{m} f_i = \bigwedge_{i=1}^{m} (r \wedge f_i) \in F$. By assumption to get that there are n of $(r \wedge f_i)'$s whose meet is in F and hence there are n of $f_i's$ whose meet is in $F \cap \langle r]$. Hence the result. $\square$

In the following, we extend Stone Stone (1938) on prime ideals to $(m,n)-$absorbing ideals in an ADL.

Theorem 3.8. *Let K be an ideal and G a non-empty subset in L such that $\bigwedge_{i=1}^{m} f_i \in G$ implies there are n of $f_i^{'}s$ whose meet is in F, for all $f_1, f_2, ..., f_m \in L$ and $K \cap G = \emptyset$. Then there exists an $(m,n)-$absorbing ideal F in L such that $K \subseteq F$ and $F \cap G = \emptyset$.*

Proof. Put $\mathcal{A} = \{H \subseteq L : H \text{ is an ideal such that } K \subseteq H \text{ and } H \cap G = \emptyset\}$. Clearly $\mathcal{A} \neq \emptyset$, since $K \in \mathcal{A}$ and hence $(\mathcal{A}, \subseteq)$ is a poset. It can be easily verified that the hypothesis of Zorn's lemma is satisfied in $(\mathcal{A}, \subseteq)$. Thus there exists a maximal member, say F in $\mathcal{A}$ such that $K \subseteq F$ and $F \cap G = \emptyset$. It is sufficient to prove F is an $(m,n)-$absorbing ideal. Since $F \cap G = \emptyset$ and $G \neq \emptyset$, it follows that $F \neq L$ and hence F is proper. Let $f_1, ..., f_m \in L$ such that there are n of $f_i's$ whose meet in F; that is, $\bigwedge_{i=1}^{n} f_i \notin F$, $\bigwedge_{i=2}^{n+1} f_i \notin F$ and so on. Then $F \vee \langle \bigwedge_{i=1}^{n} f_i]$, $F \vee \langle \bigwedge_{i=2}^{n+1} f_i]$ and so on, are ideals properly containing F. By the maximality of F, $F \vee \langle \bigwedge_{i=1}^{n} f_i]$, $F \vee \langle \bigwedge_{i=2}^{n+1} f_i]$ and so on, are not a members of $\mathcal{A}$. Thus, $\left(F \vee \langle \bigwedge_{i=1}^{n} f_i]\right) \cap G \neq \emptyset$, $\left(F \vee \langle \bigwedge_{i=2}^{n+1} f_i]\right) \cap G \neq \emptyset$ and so on. Choose $h_1 \in \left(F \vee \langle \bigwedge_{i=1}^{n} f_i]\right) \cap G$, $h_2 \in \left(F \vee \langle \bigwedge_{i=2}^{n+1} f_i]\right) \cap G$ and so on. Then $\bigwedge_{i=1}^{m} h_i \in G$ and $\bigwedge_{i=1}^{m} h_i \in \left((F \vee \langle \bigwedge_{i=1}^{n} f_i]) \wedge (F \vee \langle \bigwedge_{i=2}^{n+1} f_i]) \wedge ...\right) = F \vee \left(\langle \bigwedge_{i=1}^{n} f_i] \wedge \langle \bigwedge_{i=2}^{n+1} f_i] \wedge ...\right) = F \vee \langle \bigwedge_{i=1}^{m} f_i]$, since $F \cap G = \emptyset$. It follows that $\bigwedge_{i=1}^{m} f_i \notin F$. Thus, F is an $(m,n)-$absorbing ideal. $\square$

Corollary 3.9. *Let G be an ideal in L and F a prime ideal containing G iff, for any $a \in F$, there exist $b \notin F$ such that $a \wedge b \in F$.*

Theorem 3.10. *If F_i is an $(m_i, n_i)-$absorbing ideal in L for each $1 \leq i \leq m^*$, then $\bigcap_{i=1}^{m^*} F_i$ is an $(m,n)-$absorbing ideal, where $n = \bigvee_{i=1}^{m^*} n_i$ and $m = \bigvee_{i=1}^{m^*} m_i \vee (n+1)$.*

Proof. To prove this theorem by using mathematical induction. It is true for $m^* = 1$. That is, $\bigcap F_1$ is a $(1,1)-$absorbing ideal as F_1 is a $(m_1, n_1)-$absorbing ideal. Suppose $m^* = k$; that is, $\bigcap_{i=1}^{k} F_i$ is an $(m,n)-$absorbing ideal, where $n = \bigvee_{i=1}^{k} n_i$ and $m = \bigvee_{i=1}^{k} m_i \vee (n+1)$ is true as F_i is an $(m_i, n_i)-$absorbing ideal in L for each $1 \leq i \leq k$. We prove that for $m^* = k+1$. That is, to prove $\bigcap_{i=1}^{k+1} F_i$ is an $(m,n)-$absorbing ideal, where $n = \bigvee_{i=1}^{k+1} n_i$ and $m = \bigvee_{i=1}^{k+1} m_i \vee (n+1)$ as F_i is an $(m_i,n_i)-$absorbing ideal in L, for each $1 \le i \le k+1$. As $\bigcap_{i=1}^{k+1} F_i = \bigcap_{i=1}^{k} F_i \cap F_{k+1}$, $\bigcap_{i=1}^{k} F_i$ is an $(m,n)-$absorbing ideal, for $n = \bigvee_{i=1}^{k} n_i$ and $m = \bigvee_{i=1}^{k} m_i \vee (n+1)$ and F is an ideal, we conclude that $\bigcap_{i=1}^{k+1} F_i$ is an $(m,n)-$absorbing ideal, for $n = \bigvee_{i=1}^{k+1} n_i$ and $m = \bigvee_{i=1}^{k+1} m_i \vee (n+1)$. Thus, $\bigcap_{i=1}^{m^*} F_i$ is an $(m,n)-$absorbing ideal. □

Lemma 3.11. *A proper ideal F in L is an $(m,n)-$absorbing ideal iff whenever $\bigwedge_{i=1}^{k} f_i \in F$, for all $f_1,...,f_k \in L$ with $k \ge m$, then there are n of the $f_i's$ whose meet is in F.*

Recalling that two ideals I and J are co-maximal if $I \vee J = L$.

Theorem 3.12. *Let $G_1,...,G_m$ be prime ideals in L that are pairwise co-maximal. Then $\bigcap_{i=1}^{m} G_i$ is an $(m,n)-$absorbing ideal in L.*

Proof. Put $F = \bigcap_{i=1}^{m} G_i$. Let $G_1,...,G_m$ be prime ideals that are pairwise co-maximal. To prove that F is an $(m,n)-$absorbing ideal in L. For $m=2$, it clear; that is, every prime ideal is $(2,1)-$absorbing ideal, by 3 (1). Assume it is true for $m=k$. That is, F is a $(k,n)-$absorbing ideal and hence there are n of G_i's whose intersection is in F. To prove for $m=k+1$; that is; to prove that F is a $(k+1,n)-$absorbing ideal and hence there are n of G_i's whose intersection is in F. As, $\bigcap_{i=1}^{k+1} G_i = \bigcap_{i=1}^{k} G_i \cap G_{k+1}$, $\bigcap_{i=1}^{k} G_i$ is a $(k,n)-$absorbing ideal and F is an ideal, we conclude that $\bigcap_{i=1}^{k+1} G_i$ is a $(k+1,n)-$absorbing ideal. Thus, by using mathematical induction F is an $(m,n)-$absorbing ideal. □

Corollary 3.13. *Let $G_1,...,G_m$ be maximal ideals in L. Then $\bigcap_{i=1}^{m} G_i$ is an $(m,n)-$absorbing ideal in L.*

Theorem 3.14. *Let G be an ideal in L. Then there is an $(m,n)-$absorbing ideal in L which is minimal $(m,n)-$absorbing ideal in L containing G. In particular, L has a minimal $(m,n)-$absorbing ideal.*

Proof. Let $\mathcal{F}$ be the set of all $(m,n)-$absorbing ideals in L containing G. Since every maximal ideal in L containing G is an $(m,n)-$absorbing ideal. So, $\mathcal{F} \neq \emptyset$. It is cleat that $(\mathcal{F}, \le)$ is a partial ordered set in which $G_1 \le G_2$, for all $G_1, G_2 \in \mathcal{F}$. Let $S = \{G_\alpha\}_{\alpha\in\Delta}$ be an arbitrary non-empty chain of elements of $\mathcal{F}$ and let $K = \bigcap_{\alpha\in\Delta} G_\alpha$. We show that K is an $(m,n)-$absorbing ideal in L. Since $S \neq \emptyset$, $K \neq L$. Suppose $\bigwedge_{i=1}^{m} f_i \in K$, for some $f_1,...,f_m \in L$. Assume that there are n of the $f_i's$ whose meet is not in K. Since S is a chain, there is $G_\alpha \in S$ such that no n of f_i's whose meet is in G_α, gives a contradiction. Thus there are n of the $f_i's$ whose meet is in K. By Zorn's lemma, $(\mathcal{F}, \le)$ has a maximal element; that is, there is a minimal $(m,n)-$absorbing ideal of L containing G. □

Corollary 3.15. *Let G be an ideal in L. Then there exists a minimal $n-$absorbing ideal in L containing G.*

Corollary 3.16. *Let F be an $(m,n)-$absorbing ideal in L and $s,t \in Z^+$ such that $2 \le s \le t \le m-1$. If F has a $(t,s)-$minimal prime ideal, then F has at least $(t,s)-$minimal absorbing ideal.*

Theorem 3.17. *Let F be an $(m,n)-$absorbing ideal in L such that F has exactly $(m,n)-$*

minimal prime ideals, say $G_1,...,G_m$. *Then* $\bigcap_{i=1}^{m} G_i \subseteq F$.

Proof. Let $m=1$. Clearly 1-absorbing ideal is prime. Assume $m\geq 2$. Let $f_i\in G_i$, for each $1\leq i\leq m$. Thus $\bigwedge_{i=1}^{m} f_i \in \bigcap_{i=1}^{m} G_i$. As $\bigwedge_{i=1}^{m+1} f_i \in F$ and F is an ideal, we conclude that $\bigwedge_{i=1}^{m} f_i\in F$. Hence the result. □

Corollary 3.18. *Let F be an $(m,n)-$absorbing ideal in L such that F has exactly $(m,n)-$minimal prime ideals, say $G_1,...,G_m$. If G_i's are co-maximal, then* $\bigcap_{i=1}^{m} G_i = F$.

Next, we introduce the notion of the direct product of $(m,n)-$absorbing ideal in $L_1\times L_2$, where L_1 and L_1 are ADLs. Let F and G be ideals in L_1 and L_2, respectively. Let $(a,b),(c,d)\in F$. Then $(a,b)\vee(c,d)=(a\vee c, b\vee d)\in F\times L_2$, since $a\vee c\in F$. Also, $(a,b)\wedge(r,s)=(a\wedge r, b\wedge s)\in F\times L_2$, since $a\wedge r\in F$. Thus $F\times L_2$ is an ideal. Similarly, $L_1\times G$ is an ideal. In the case of $(m,n)-$absorbing ideal, we have the following.

Theorem 3.19. *Let $L=L_1\times L_2$. Then the following assertion hold. If F is an $(m,n)-$absorbing ideal in L_1, then $F\times L_2$ is an $(m,n)-$absorbing ideal in L. Also, if G is an $(m,n)-$absorbing ideal in L_2, then $L_1\times G$ is an $(m,n)-$absorbing ideal in L.*

Proof. Suppose F is an $(m,n)-$absorbing ideal in L_1. Let $f_1,f_2,...,f_m\in L_1$ such that $\bigwedge_{i=1}^{m}(f_i,r)\in F\times L_2$, for some $r\in L_2$. As there are n of the f_i's whose meet is in F if $\bigwedge_{i=1}^{m} f_i\in F$. Thus, there are n of the $(f_i,r)'$s whose meet is in $F\times L_2$. Therefore, $F\times L_2$ is an $(m,n)-$absorbing ideal in L. Similarly, $L_1\times G$ is an $(m,n)-$absorbing ideal in L if G is an $(m,n)-$absorbing ideal in L_2. □

In the following, we establish that both the image and pre-image of any $(m,n)-$absorbing ideal is again $(m,n)-$absorbing ideal.

Theorem 3.20. *Let L_1 and L_2 be ADLs and $h: L_1\to L_2$ be a lattice homomorphism. Then the following hold.*
(1). Let G be an $(m,n)-$absorbing ideal in L_2. Then $h^{-1}(G)$ is an $(m,n)-$absorbing ideal in L_1
(2). Let h be an epimorphism and F be an ideal. Then F is an $(m,n)-$absorbing ideal in L_1 containing $ker(h)$ iff $h(F)$ is an $(m,n)-$absorbing ideal in L_2.

Proof. (1). Suppose G is an $(m,n)-$absorbing ideal in L_2. Let $f_1,...,f_m\in L_1$ such that $\bigwedge_{i=1}^{m} f_i\in h^{-1}(G)$. Then $h(\bigwedge_{i=1}^{m} f_i)=\bigwedge_{i=1}^{m} h(f_i)\in G$, and hence there are n of f_i's whose meet is in $h^{-1}(G)$, since there are n of $h(f_i)'$s whose meet is in G. Thus, $h^{-1}(G)$ is an $(m,n)-$absorbing ideal in L_1.
(2). Suppose F is an $(m,n)-$absorbing ideal in L_1. Let $g_1,g_2,...,g_m\in L_2$ such that $h(f_1)=g_1$, $h(f_2)=g_2,...,h(f_m)=g_m$, for some $f_1,f_2,...,f_m\in L_1$. Since $ker(h)\subseteq F$, then $h(F)$ is proper. Suppose that $\bigwedge_{i=1}^{m} g_i\in h(F)$. Since $\bigwedge_{i=1}^{m} g_i\in h^{-1}(h(F))$, $\bigwedge_{i=1}^{m} h(f_i)=\bigwedge_{i=1}^{m} g_i\in h(F)$ and there are n of f_i's whose meet is in F, we conclude that there are n of g_i's whose meet is in $h(F)$. Hence the result. □

4 Weakly $(m,n)-$Absorbing Ideal

In this section, we introduce the concepts of weakly $(m,n)-$absorbing ideal, generalize the notion of weakly prime ideals and $(m,n)-$absorbing ideals. We justify several properties and characterizations of weakly $(m,n)-$absorbing ideals with supportive examples. Furthermore, we investigate the direct product, homomorphic images and pre-images of weakly $(m,n)-$absorbing ideals.

Definition 4.1. *Let $m, n \in Z^+$ with $m > n$. A proper ideal F in L is a weakly $(m,n)-$absorbing ideal in L if for all $f_1, f_2, ..., f_m \in L$ such that $0 \neq \bigwedge_{i=1}^{m} f_i \in F$, there are n of $f_i^{'}s$ whose meet is in F.*

In the following, we introduce the relationship between $(m,n)-$absorbing ideal and weakly $(m,n)-$absorbing ideal.

Theorem 4.2. *Every $(m,n)-$absorbing ideal is a weakly $(m,n)-$absorbing ideal and the converse of this is not true.*

Example 4.3. *Let $C = \{0, u, v\}$ be a discrete ADL with 0 as its zero element defined in 2.2 and $D = \{0, a, b, c, d, e, f, g, h, i, j, 1\}$ be a lattice defined in 3.6. Put $I = \{(0,0)\}$, is an ideal in $C \times D$. Clearly I is a weakly $(4,2)-$absorbing ideal in $C \times D$ but not an $(4,2)-$absorbing ideal, since $(u,d) \wedge (v,e) \wedge (0,f) \wedge (u,g) \in I$ implies $(u,d) \wedge (v,e) = (u,a) \notin I$, $(u,d) \wedge (0,f) = (0,b) \notin I$, $(u,d) \wedge (u,g) = (u,0) \notin I$, $(v,e) \wedge (0,f) = (0,c) \notin I$, $(v,e) \wedge (u,g) = (u,c) \notin I$ and $(0,f) \wedge (u,g) = (0,c) \notin I$.*

Next, we characterize weakly $(m,n)-$absorbing ideals in direct product of ADLs.

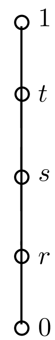
Corollary 4.4. *Let $F(\neq \{0\})$ be a proper ideal in $L = L_1 \times L_2$. Then F is a weakly $(m,n)-$absorbing ideal in L iff F is an $(m,n)-$absorbing ideal in L.*

Theorem 4.5. *Let F and G be proper ideals of L_1 and L_2. If $F \times G$ is a weakly $(m,n)-$absorbing ideal in $L_1 \times L_2$, then F are G are weakly $(m,n)-$absorbing ideals in L_1 and L_2, respectively.*

Proof. Suppose $F \times G$ is a weakly $(m,n)-$absorbing ideal in $L_1 \times L_2$. Let $f_1, f_2, ..., f_m \in L_1$ such that $0 \neq \bigwedge_{i=1}^{m} f_i \in F$ and $0 \neq \bigwedge_{i=1}^{m} g_i \in G$, for all $g_1, g_2, ..., g_m \in L_2$. Then $(0,0) \neq (\bigwedge_{i=1}^{m} f_i, \bigwedge_{i=1}^{m} g_i) \in F \times G$. By assumption to get that, there are n of the $(f_i, g_i)'$s whose meet is in $F \times G$ and hence there are n of f_i's and g_i's whose meet are in F and G, respectively. Thus, F are G are weakly $(m,n)-$absorbing ideals in L_1 and L_2, respectively. □

If there are weakly $(m,n)-$absorbing ideals, then their direct product may not weakly $(m,n)-$absorbing ideal; consider the following example.

Example 4.6. *Let $L_1 = \{0, a, b, c, d, e, f, 1\}$ be a lattice defined in 3.2 and $L_2 = \{0, r, s, t, 1\}$ be a lattice represented by the Hasse diagram given below:*



Consider $L_1 \times L_2 = \{(x,y) : x \in L_1$ and $y \in L_2\}$. Put $F = \{0\}$ and $G = \{0, r\}$. Clearly F and G are weakly $(3,2)-$absorbing ideals in L_1 and L_2, respectively. But $F \times G = \{(0,0), (0,r)\}$ is not a weakly $(3,2)-$absorbing ideal in $L_1 \times L_2$, since $(0,0) \neq (d,1) \wedge (e,t) \wedge (f,r) \in F \times G$ implies $(d,1) \wedge (e,t) = (a,t) \notin F \times G$, $(d,1) \wedge (f,r) = (b,r) \notin F \times G$, and $(e,t) \wedge (f,r) = (c,r) \notin F \times G$.

Theorem 4.7. *Let L_1 and L_2 be ADLs and $F(\neq \{0\})$ be a proper ideal in L_1. Then the following are equivalent.*
(1). $F \times L_2$ is a weakly $(m,n)-$absorbing ideal in $L_1 \times L_2$
(2). $F \times L_2$ is an $(m,n)-$absorbing ideal in $L_1 \times L_2$
(3). F is an $(m,n)-$absorbing ideal in L_1.

Proof. $(1) \Leftrightarrow (2)$: It is clear.
$(2) \Leftrightarrow (3)$: Assume (2) hold. Let

$f_1, f_2, ..., f_m \in L_1$ with $\bigwedge_{i=1}^{m} f_i \in F$. Clearly $(\bigwedge_{i=1}^{m} f_i, r) \in F \times L_2$, for some $r \in L_2$. By (2), we have there are n of $(f_i, r)'$s whose meet is in $F \times L_2$ and hence there are n of f_i's whose meet is in F. Hence the result. The converse is clear.
(3) $\Leftrightarrow$ (1) : Suppose F is an $(m,n)-$absorbing ideal in L_1. Let $f_1, f_2, ..., f_m \in L_1$ such that $(0,0) \neq (\bigwedge_{i=1}^{m} f_i, r) \in F \times L_2$, for some $r \in L_2$. By assumption, there are n of f_i's whose meet is in F and hence there are n of $(f_i, r)'$s whose meet is in $F \times L_2$. Thus, $F \times L_2$ is a weakly $(m,n)-$absorbing ideal in $L_1 \times L_2$. The converse is clear. □

The following Theorem is an immediate consequence of 4.5 and 4.7.

Corollary 4.8. *Let $F(\neq \{0\})$ and $G(\neq \{0\})$ be proper ideals in L_1 and L_2, respectively. Then the following are equivalent.*
(1). $F \times G$ is a weakly $(m,n)-$absorbing ideal in $L_1 \times L_2$
(2). $G = L_2$ and F is an $(m,n)-$absorbing ideal in L_1, or G is an $(m,n)-$prime ideal in L_2 and F is an $(m,n)-$prime ideal ideal in L_1
(3). $F \times G$ is an $(m,n)-$absorbing ideal in $L_1 \times L_2$.

Corollary 4.9. *Let $L = L_1 \times L_2 \times ... \times L_k$ and $H(\neq \{0\})$ be proper ideal in L. Then the following are equivalent.*
(1). F is a weakly $(m,n)-$absorbing ideal in L
(2). $F = L_1 \times L_2 \times ... \times F_j \times ... \times L_k$, where F_j is an $(m,n)-$absorbing ideal in L_j, for some $j \in \{1, 2, ..., k\}$
(3). F is an $(m,n)-$absorbing ideal in L.

Finally, we discuss the homomorphism of weakly $(m,n)-$absorbing ideals.

Theorem 4.10. *Let L_1 and L_2 be ADLs and $h : L_1 \to L_2$ be a lattice homomorphism. If k is a monomorphism and G is a weakly $(m,n)-$absorbing ideal in L_2, then $h^{-1}(G)$ is a weakly $(m,n)-$absorbing ideal in L_1. Also, let h be an epimorphism. Then F is a weakly $(m,n)-$absorbing ideal in L_1 containing $ker(k)$ iff $h(F)$ is a weakly $(m,n)-$absorbing ideal in L_2.*

5 Conclusion

We define the notions of $(m,n)-$absorbing ideals and weakly $(m,n)-$absorbing ideals in an ADL and discuss their properties. Also, we introduce the concept of weakly $(m,n)-$absorbing ideals, generalizing weakly prime ideals and $(m,n)-$absorbing prime ideals. Furthermore, we explore the properties of $(m,n)-$absorbing ideals and weakly $(m,n)-$absorbing ideals for various lattice-theoretic construction such as direct products, homomorphism images, and homomorphic inverse images. In future work, we plan to focus on the concepts of L-fuzzy $(m,n)-$absorbing ideals and their prime spectrum.

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