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Original Research Article

Soft Set Theory Applied to MS-Algebras

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ABSTRACT

This study provides a foundational investigation of MS-algebras through the lens of soft set theory. The paper introduces and analyzes the notions of soft MS-algebras, soft e-filters, and soft MS-homomorphisms, along with their fundamental properties and interrelationships.

Keywords: Soft sets, MS-algebras, Soft MS-algebras, e-filters, Soft e-filters

1 Introduction

Molodtsov first introduced the idea of soft sets as a novel mathematical framework for addressing problems involving uncertainty. In many complex areas such as economics, engineering, environmental studies, medicine, and social sciences, traditional mathematical techniques often fail to provide adequate solutions due to the presence of uncertainty. To manage such issues, researchers have developed theories like probability theory, fuzzy set theory, and rough set theory. However, each of these approaches has its own limitations. To overcome these shortcomings, Molodtsov (1999) proposed the concept of soft sets as an alternative tool for handling uncertainty. Unlike fuzzy sets, soft set theory does not require the construction of membership functions, which makes it more flexible and easier to apply across various fields.

Subsequent studies expanded on this foundation: Maji et al. (2003) investigated certain operations on soft sets, while Irfan Ali et al. (2009) introduced additional operations. Aktas and Cagman (2007) compared soft sets with fuzzy and rough sets, and further developed the notion of soft groups along with their properties. Nagarajan and Meenambigai (2011) explored structures such as soft lattices, soft distributive lattices, and soft modular lattices.

In this work, we extend these ideas by defining the concept of soft MS-algebras and examining their properties, supported with several illustrative examples. We also introduce the notion of homomorphisms between soft MS-algebras and study their characteristics. Furthermore, we define soft e-filters of MS-algebras and provide examples to clarify their use. All results presented in this paper are derived using Molodtsov's original framework of soft sets.

For an in-depth discussion of the general framework of MS-algebras, the reader is referred to Blyth et al. (1988). The definitions and fundamental properties of filters and (e-)-filters in MS-algebras are adopted from roa et. al (2002), while the theoretical background of soft sets follows the approaches presented in Molodtsov (1999) and maj et .al (2003). Unless otherwise indicated, M denotes an MS-algebra throughout this section.

2 Soft MS-Algebras

In this section, we introduce the concept of soft MS-algebras and investigate several of their fundamental properties. Throughout, let M denote an MS-algebra and A be a nonempty set. Let R be a binary relation between the elements of A and those of M. Define a set-valued mapping $K : A \to P(M)$ by

$$K(t) = \{s \in M : tRs,\, t \in A\}.$$

Then, the ordered pair (K, A) forms a soft set over M. Conversely, each such mapping K determines a binary relation R on $A \times M$ given by

$$R = \{(t, s) \in A \times M : s \in K(t)\}.$$

Definition 2.1. Let (K, A) be a soft set over M. Then (K, A) is called a *soft MS-algebra over* M if, for every $t \in A$, the set $K(t)$ is a sub-MS-algebra of M.

The family of all soft MS-algebras over M is denoted by $\mathcal{S}_{\text{MS}}(M)$. We now illustrate this concept with a series of examples.

Example 2.2. *Let M be the MS-algebra depicted in Figure 1, and let $A = M$. Define the mapping $K : A \to P(M)$ by*

$$K(p) = \{q \in M : pRq \Leftrightarrow p \wedge q^\circ \wedge q^{\circ\circ} = 0,\, p \in M\}.$$

From this definition, we obtain

$$K(0) = M, \qquad K(t) = K(p) = K(1) = K(z) = K(u) = K(y) = \{0, 1\}.$$

Hence, for each $p \in A$, $K(p)$ is a sub-MS-algebra of M, and consequently $(K, A) \in \mathcal{S}_{MS}(M)$.

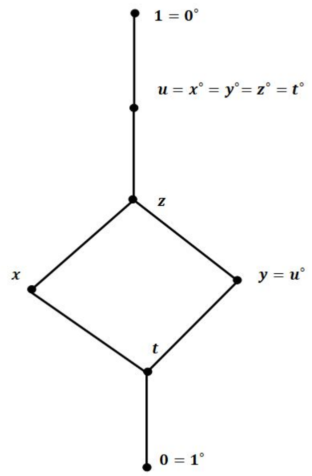

Figure-1

Example 2.3. *Consider again the MS-algebra M shown in Figure 1 and take $A = M$. Define $K : A \to P(M)$ by*

$$K(p) = \{q \in M : pRq \Leftrightarrow p \wedge q^\circ \wedge q^{\circ\circ} = 0,\, p \in M\}.$$

Then

$$K(0) = M, \qquad K(t) = K(p) = K(1) = K(z) = K(u) = K(y) = \{0, 1\}.$$

Therefore, $(K, A) \in \mathcal{S}_{MS}(M)$.

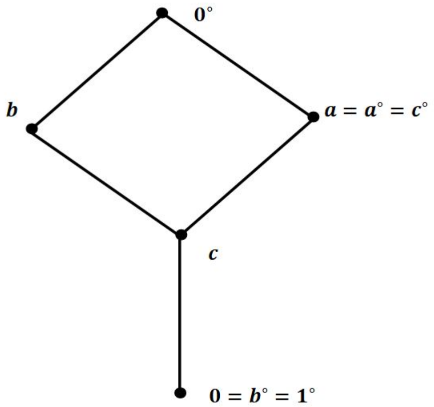

Figure-2

Example 2.4. *Let M be the MS-algebra given in Figure 2 and set $B = \{0, 1\}$. Define*

$$K(0) = M, \quad K(1) = \{0, 1\}.$$

Then (K, B) is a soft MS-algebra over M.

Remark 2.5. Every MS-algebra can naturally be regarded as a soft MS-algebra.

Theorem 2.6. *If $(K, A), (L, B) \in \mathcal{S}_{MS}(M)$, then their intersection*

$$(K, A) \cap (L, B) = (M, C),$$

where $C = A \cap B \neq \emptyset$ and $M(p) = K(p) \cap L(p)$ for all $p \in C$, also belongs to $\mathcal{S}_{MS}(M)$.

Proof. For any $p \in C = A \cap B$, $K(p)$ and $L(p)$ are sub-MS-algebras of M. Their intersection $M(p) = K(p) \cap L(p)$ is therefore a sub-MS-algebra of M. Hence $(M, C) \in \mathcal{S}_{\text{MS}}(M)$. □

Theorem 2.7. *Let $(K, C), (L, D) \in \mathcal{S}_{MS}(M)$. If $C \cap D = \emptyset$, then*

$$(K, C) \cup (L, D) \in \mathcal{S}_{MS}(M).$$

Proof. The union of (K, C) and (L, D) is (M, E), where $E = C \cup D$ and

$$M(q) = \begin{cases} K(q), & q \in C \setminus D, \\ L(q), & q \in D \setminus C, \\ K(q) \cup L(q), & q \in C \cap D. \end{cases}$$

Since $C \cap D = \emptyset$, it follows that $M(q) = K(q)$ for $q \in C$ and $M(q) = L(q)$ for $q \in D$. As each of these is a sub-MS-algebra of M, we have $(M, E) \in \mathcal{S}_{\text{MS}}(M)$. □

Theorem 2.8. *Let* $(K, A), (L, B) \in \mathcal{S}_{\text{MS}}(M)$ *such that* $K(p) \cap L(q) \neq \emptyset$ *for all* $p \in A$ *and* $q \in B$. *Then the AND operation satisfies*

$$(K, A) \wedge (L, B) \in \mathcal{S}_{\text{MS}}(M).$$

Proof. Define $(K, A) \wedge (L, B) = (M, C)$, where $C = A \times B$ and $M((p, q)) = K(p) \cap L(q)$ for all $(p, q) \in C$. Since the intersection of two sub-MS-algebras is a sub-MS-algebra, we obtain $(M, C) \in \mathcal{S}_{\text{MS}}(M)$. □

Definition 2.9. Let (K, A) and (L, B) be soft MS-algebras over M. Then (L, B) is called a *soft subMS-algebra* of (K, A) if

1. $B \subseteq A$, and
2. $L(x)$ is a sub-MS-algebra of $K(x)$ for all $x \in B$.

Example 2.10. *Consider the MS-algebra* M *in Figure 3. Let* $A = M$, $B = \{0, 1, a, b\}$, *and define* $K : A \to P(M)$ *by*

$$K(p) = \{q \in M : pRq \Leftrightarrow p \vee q^\circ \vee q^{\circ\circ} = 1\}.$$

Then

$$K(0) = \{0, 1\}, \qquad K(1) = M,$$
$$K(a) = K(c) = \{0, 1, c, d, b\}, \quad K(b) = \{0, 1, c, a\}, \quad K(d) = \{0, 1, c\}.$$

Now define $L : B \to P(M)$ *by*

$$L(p) = \{q \in M : pRq \Leftrightarrow p \vee q^\circ \vee q^{\circ\circ} = 1\}, \quad p \in B,$$

which gives

$$L(0) = \{0, 1\}, \quad L(1) = M, \quad L(a) = \{0, 1, c, d, b\}, \quad L(b) = \{0, 1, c, a\}.$$

Thus, $(K, A), (L, B) \in \mathcal{S}_{\text{MS}}(M)$, *and since* $B \subseteq A$ *and each* $L(p)$ *is a sub-MS-algebra of* $K(p)$, (L, B) *is a soft subMS-algebra of* (K, A).

Next, define $M : A \to P(M)$ *by*

$$M(p) = \{q \in M : pRq \Leftrightarrow p \wedge q^\circ \wedge q^{\circ\circ} = 0\}, \quad p \in A.$$

Then

$$M(0) = M, \quad M(1) = \{0, 1, c\}, \quad M(a) = \{0, 1, c, d, b\}.$$

Although $(K, A), (M, A) \in \mathcal{S}_{\text{MS}}(M)$ *and* $A \subseteq A$*, we have* $M(0)$ *is not a subset of* $K(0)$*; hence* (M, A) *is not a soft subMS-algebra of* (K, A)*.*

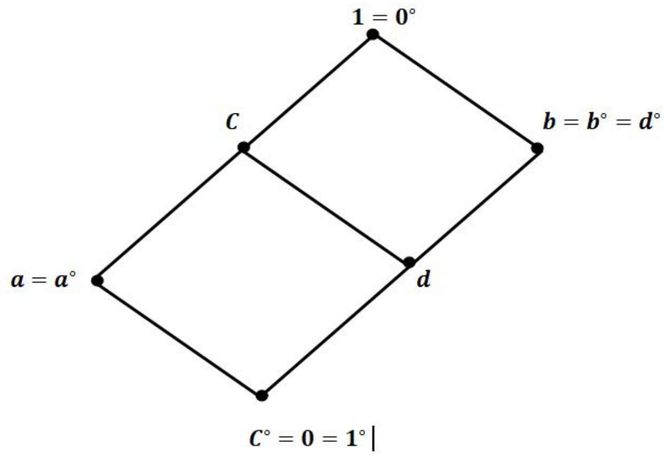

Figure-3

Theorem 2.11. *Let* $(K, A), (L, A) \in \mathcal{S}_{\text{MS}}(M)$*. Then* (K, A) *is a soft subMS-algebra of* (L, A) *if and only if*

$$K(p) \subseteq L(p), \quad \forall p \in A.$$

Proof. If (K, A) is a soft subMS-algebra of (L, A), then by definition $K(p) \subseteq L(p)$ for all $p \in A$. Conversely, if $K(p) \subseteq L(p)$ for all $p \in A$, then each $K(p)$ is a sub-MS-algebra of $L(p)$, and thus (K, A) is a soft subMS-algebra of (L, A). □

Corollary 2.12. *Every soft MS-algebra is a soft subMS-algebra of itself; that is,* (K, A) *is a soft subMS-algebra of* (K, A)*.*

Definition 2.13. Let f be an MS-algebra homomorphism from L_1 to L_2, and let (K, A) be a soft MS-algebra over L_1. The *image* of (K, A) under f is the soft set $(f(K), A)$ defined by

$$(f(K))(p) = f(K(p)), \quad \forall p \in A.$$

Theorem 2.14. *Let* (K, A) *and* (L, B) *be soft MS-algebras over* L_1*, with* (K, A) *a soft subMS-algebra of* (L, B)*. If* $f : L_1 \to L_2$ *is a homomorphism, then* $(f(K), A)$ *is a soft subMS-algebra of* $(f(L), B)$*.*

Proof. Since (K, A) is a soft subMS-algebra of (L, B), we have $A \subseteq B$ and $K(p)$ a sub-MS-algebra of $L(p)$ for all $p \in A$. The image of a sub-MS-algebra under a homomorphism is a sub-MS-algebra; hence $f(K(p))$ is a sub-MS-algebra of $f(L(p))$ for all $p \in A$. Therefore, $(f(K), A)$ is a soft subMS-algebra of $(f(L), B)$. □

Definition 2.15. Let L_1 and L_2 be MS-algebras with unary operation $(\cdot)^\circ$. Then the Cartesian product $L_1 \times L_2$ forms an MS-algebra under the pointwise operation

$$(p, q)^\circ = (p^\circ, q^\circ).$$

Definition 2.16. Let (K, A) and (L, B) be soft MS-algebras over M_1 and M_2, respectively. Their *product* is the soft set $(K, A) \times (L, B) = (M, A \times B)$, where

$$M(p, q) = K(p) \times L(q), \quad \forall (p, q) \in A \times B.$$

Theorem 2.17. *If (K, A) and (L, B) are soft MS-algebras over M_1 and M_2, respectively, then their product $(M, A \times B)$ is a soft MS-algebra over $M_1 \times M_2$.*

Proof. For any $(p, q) \in A \times B$, since $K(p)$ and $L(q)$ are sub-MS-algebras of M_1 and M_2, respectively, the Cartesian product $K(p) \times L(q)$ forms a sub-MS-algebra of $M_1 \times M_2$. Hence, $(M, A \times B) \in \mathcal{S}_{\text{MS}}(M_1 \times M_2)$. □

Definition 2.18. Consider two soft MS-algebras (K, C) and (L, D) defined over M_1 and M_2, respectively. A pair of functions (f, g), where $f : M_1 \to M_2$ and $g : C \to D$, is said to be a *soft MS-algebra homomorphism* if the following conditions hold:

1. f acts as a homomorphism of MS-algebras from M_1 onto M_2,
2. g is a surjective function mapping C onto D, and
3. for every $x \in C$, the equality $f(K(x)) = L(g(x))$ holds.

In such a situation, (K, C) is referred to as being *homomorphic* to (L, D), which is expressed as $(K, C) \sim (L, D)$. Furthermore, if f is an isomorphism and g is a bijection, then (f, g) is called a *soft MS-algebra isomorphism*, symbolized by $(K, C) \simeq (L, D)$.

Example 2.19. *Consider the MS-algebras L_1 and L_2 shown in Figures 4 and 5, respectively. Let $A = \{0, a, b, 1\}$ and $B = \{0', 1'\}$. Define the mapping $K : A \to P(L_1)$ by*

$$K(p) = \{q \in L_1 : pRq \Leftrightarrow p \wedge q^\circ \wedge q^{\circ\circ} = 0,\, p \in A\},$$

so that $(K, A) \in \mathcal{S}_{\text{MS}}(L_1)$*. Similarly, define* $M : B \to P(L_2)$ *by*

$$M(p) = \{q \in L_2 : pRq \Leftrightarrow p \wedge q^\circ \wedge q^{\circ\circ} = 0,\, p \in B\},$$

yielding $(M, B) \in \mathcal{S}_{\text{MS}}(L_2)$*.*

Now define $f : L_1 \to L_2$ *and* $g : A \to B$ *by*

$$f(0) = 0', \quad f(a) = 0', \quad f(b) = 1', \quad f(1) = 1',$$

$$g(0) = 0', \quad g(a) = 0', \quad g(b) = 1', \quad g(1) = 1'.$$

Here, f *is an MS-algebra homomorphism from* L_1 *onto* L_2*, and* g *is surjective from* A *to* B*. Furthermore,*

$$f(K(p)) = M(g(p)), \quad \forall p \in A.$$

Hence, (K, A) *is a soft MS-algebra homomorphic to* (M, B)*.*

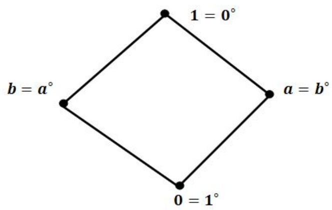

Figure-4

Figure-5

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