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ORIGINAL RESEARCH

δ-ideals in MS-Almost Distributive Lattices

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ABSTRACT

In this paper, we introduce the concept of δ-ideals in MS-ADL analogous to that of MS-algebras and explores their properties. It is proved that the class of all δ-ideals forms a complete distributive lattice. We provided a set of equivalent conditions for a given ideal of an MS-ADL to be a δ-ideal. Furthermore, the image and inverse image of δ-ideals also studied under a homomorphism mapping.

Keywords/phrases: MS-algebras, almost distributive lattices, δ-ideals, MS-ADLs, homomorphisms MSC Code: 06D99, 06D15

Introduction

The concept of an almost distributive lattice (ADL) was introduced by Swamy and Rao (1981) as a common abstraction of most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. An ADL is an algebra with two binary operations "∨" and "∧" which satisfies most of the properties of a distributive lattice with smallest element 0, except possibly the commutativity of the binary operations "∨" and "∧" , and the right distributivity of "∨" over "∧". It was also observed that any one of these three properties converts an ADL into a distributive lattice. The class of ADLs with pseudo-complementation was introduced by Swamy *et al.* (2000). Later on, Swamy *et al.* (2003) introduced a more general class of ADLs called Stone ADLs, which properly contains the class of pseudo-complemented ADLs, as a generalization of Stone lattices. Rafi *et al.* (2014), Studied Dominator and Closure ideals in Almost Distributive Lattices. Also, Rafi et al. (2016) presented δ-ideals in pseudo-complemented almost distributive Lattices.

An Ockham algebra is a bounded distributive lattice with a dual endomorphism. The class of all Ockham algebras contains the well-known classes for example Boolean algebras, De Morgan algebras, Kleene algebras and Stone algebras (Blyth and Varlet, 1994). Berman (1977), initiated the study of a variety K of bounded distributive lattices endowed with a dual homomorphic operation paying particular attention to certain subvarieties K(p,q) for p ≥, q > 0. A subclass of Ockham algebras so-called MS-algebras was introduced by Blyth and Varlet (1983a) as a common abstraction of De Morgan algebras and Stone algebras.

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Introduction

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This class of Ockham algebras is characterized by the fact that the dual endomorphism f satisfies $f^0 \leq f^2$, which implies that $f^0 = f^2$. The class of all MS-algebras forms an equational class. Blyth and Varlet characterized the subvarieties of MS-algebras in Blyth et al. (1997). More recently, Addis (2020a), investigated a new equational class of algebras called De Morgan almost distributive lattices (De Morgan ADLs) as a common abstraction of De Morgan algebras and almost Boolean algebras (relatively complemented ADLs). He also studied another equational class of algebras called MS-almost distributive lattices (MS-ADLs) as a common abstraction of De Morgan ADLs and Stone ADLs Addis (2020b). In this paper we studied the concept of δ-ideals in MS-ADL analogous to that of MS-algebras and explores their properties. The class of all δ-ideals of MS-ADL as well as we have shown that this class forms a complete distributive lattice. Furthermore, we studied the principal δ-ideals and their properties. We provided a set of equivalent conditions for a given ideal of an MS-ADL to be a δ-ideal. Moreover, the image and inverse image of δ-ideals also studied under a homomorphism mapping.

2Preliminaries

In this section ,we recall some definitions and results which will be used in this paper.

Definition 2.1. Swamy and Rao (1981) An algebra $L = (L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL), if it satisfies the following conditions for all a, b and $c \in L$:

(1) $0 \wedge a = 0$,

(2) $a \vee 0 = a$,

(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(4) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

(5) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,

(6) $(a \vee b) \wedge b = b$.

For any $a, b \in L$, we say that a is less than or equals to b and we write $a \leq b$ if $a \wedge b = a$, equivalently $a \vee b = b$. An element m in$L$ is said to be maximal if it is maximal with respect to the partial ordering $\leq$ on L.

Lemma 2.2. *Swamy et al. (2000) Let L be an ADL and $m \in L$. Then the following are equivalent:*

(1) *m is maximal with respect to $\leq$,*

(2) *$m \vee a = m$, for all $a \in L$,*

(3) *$m \wedge a = a$, for all $a \in L$,*

(4) *$a \vee m$ is maximal, for all $a \in L$.*

Definition 2.3. Swamy and Rao (1981) A nonempty subset I of L is called an ideal (respectively a filter) of L, if $a \vee b$, $a \wedge x \in I$(respectively $a \wedge b, x \vee a \in F$) or all $a, b \in I(F)$ and all $x \in L$.

Definition 2.4. Blyth and Varlet (1983a) An MS-algebra is an algebra $(L, \vee, \wedge, ^\circ, 0, 1)$ of type $(2, 2, 1, 0, 0)$, such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a \to a^\circ$ is a unary operation satisfies: $a \leq a^{\circ\circ}$, $(a \wedge b)^\circ = a^\circ \vee b^\circ$, $1^\circ = 0$

Definition 2.5. Addis (2020b) An MS-almost distributive lattice (MS-ADL) is an algebra $(L, \vee, \wedge, ^\circ, 0)$ of type $(2, 2, 1, 0)$ such that $(L, \vee, \wedge, 0)$ is an ADL with maximal elements and $x \mapsto x^\circ$ is a unary operation on L satisfying the following axioms:

(1) $x^{\circ\circ} \wedge x = x$,

(2) $(x \vee y)^\circ = x^\circ \wedge y^\circ$,

(3) $(x \wedge y)\circ = x^\circ \vee y^\circ$,

(4) $m^\circ = 0$ for all maximal elements m of L,for all $x, y \in L$. In addition, if it satisfies the following condition:

(5) $x^{\circ\circ} = x \wedge m$, then L is called a De Morgan ADL.

Lemma 2.6. *Addis (2020b) The following holds in an MS-ADL.*

(1) 0° *is maximal,*

(2) $a \leq b \Rightarrow b^\circ \leq a^\circ$,

(3) $a^{\circ\circ\circ} = a^\circ$,

(4) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,

(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$,

(6) $(a \wedge m)^\circ = a^\circ$,

(7) $(a \wedge b)^\circ = (b \wedge a)^\circ$ *for all* $a, b \in L$.

3 δ-IDEALS OF MS-ADL

The concept of δ-ideals of MS-algebras was given by Badawy et al. (2024). In this section, we introduce the concept of δ-ideals of MS-ADL, analogously. Though many results look similar, the proofs are not similar because of the lack of the properties like: commutativity of "$\vee$", commutativity of "$\wedge$" and the right distributivity of "$\vee$" over "$\wedge$" in an ADL.

Throughout this section L stands for an MS-almost distributive lattice unless otherwise mentioned.

Definition 3.1. For any filter F of L, define

$$\delta(F) = \{x \in L : x^\circ \in F\}$$

.

Now we have the following results.

Lemma 3.2. *For any filter F of L, $\delta(F)$ is an ideal of L.*

Proof. Since $0^\circ \in F, 0 \in \delta(F)$. Then $\delta(F) \neq \emptyset$. Let $x, y \in \delta(F)$. Then $x^\circ, y^\circ \in F$. This implies $x^\circ \wedge y^\circ = (x \vee y)^\circ \in F$. This implies $x \vee y \in \delta(F)$. Let $x \in \delta(F)$ and $r \in L$. Then $x^\circ \in \delta(F)$. This implies $(x \wedge r)^\circ = (r \wedge x)^\circ = r^\circ \vee x^\circ \in F$. Hence $x \wedge r \in \delta(F)$. Therefore $\delta(F)$ is an ideal of L. □

Lemma 3.3. *For any two filters of F and G of L, we have the following:*

(1) $F \cap \delta(F) = \emptyset$, *whenever L is a stone ADL and F is a proper filter,*

(2) $x \in \delta(F)$ *implies* $x^{\circ\circ} \in \delta(F)$,

(3) $x \in F$ *implies* $x^\circ \in \delta(F)$,

(4) $F = L$ *if and only if* $\delta(F) = L$,

(5) $F \subseteq G$ *implies* $\delta(F) \subseteq \delta(G)$,

(6) $\delta(D) = \{0\}$,

(7) $\delta(F)$ *is a prime, whenever F is a prime filter of L,*

(8) $\delta(F \cap G) = \delta(F) \cap \delta(G)$.

Proof. (1) Suppose $F \cap \delta(F) \neq \emptyset$. Then there is x in $F \cap \delta(F)$. Which implies $x \in F$ and $x^\circ \in F$. Since F is a filter and L is a stone ADL, we have $x \wedge x^\circ = 0 \in F$. Which is a contradiction. Thus $F \cap \delta(F) = \emptyset$.

(2) Let $x \in \delta(F)$, then $x^{\circ\circ\circ} = x^\circ \in F$. This implies $x^{\circ\circ} \in \delta(F)$.

(3) Let $x \in F$, then $x = x^{\circ\circ} \wedge x \in F$. This implies $x^{\circ\circ} \in F$. Hence $x^\circ \in \delta(F)$.

(4) Suppose that $F = L$. Then we can choose $0 = 0^{\circ\circ} \in F$. This implies $m = 0^\circ \in \delta(F)$. This implies $\delta(F) = L$. Conversely, suppose that $\delta(F) = L$. This implies $m \in \delta(F)$ for any maximal element of m of L. This implies $m^\circ = 0 \in F$. Hence $F = L$.

(5) Suppose that $F \subseteq G$. Let $x \in \delta(F)$. Then $x^\circ \in F \subseteq G$. This implies $x \in \delta(G)$. Hence $\delta(F) \subseteq \delta(G)$.

(6) Let $x \in \delta(D)$. Then $x^\circ \in D$. This implies $x^{\circ\circ} = 0$. This implies $x = x^{\circ\circ} \wedge x = 0 \wedge x = 0$. Hence $\delta(D) = \{0\}$.

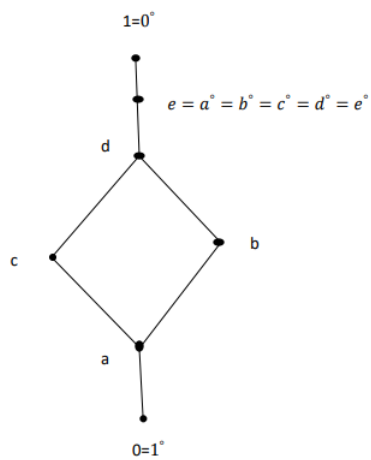
(7) Suppose F is prime filter of L. Let $x \wedge y \in \delta(F)$. Then $(x \wedge y)^\circ = x^\circ \vee y^\circ \in F$. This implies $x^\circ \in F$ or $y^\circ \in F$. This implies $x \in \delta(F)$ or $y \in \delta(F)$. Hence $\delta(F)$ is prime ideal of L. □

The concept of δ-ideal is introduced in the following.

Definition 3.4. An ideal I of L is called a δ-Ideal if $I = \delta(F)$ for some filter F of L.

Example 3.5. *Consider a discrete MS-ADL $A = \{0^{'}, a^{'}\}$ and an MS-algebra*

$B = \{0.a.b, c, d, e, 1\}$ whose Hasse diagram is given in the following Figure-1.



Take $L = A \times B = \{(0^{'}, 0), (0^{'}, a), (0^{'}, b), (0^{'}, c), (0^{'}, d), (0^{'}, e), (0^{'}, 1), (a^{'}, 0), (a^{'}, a), (a^{'}, b), (a^{'}, c), (a^{'}, d), (a^{'}, e), (a^{'}, 1)\}$. Then $(L, \vee, \wedge, ^\circ, \overline{0})$ is an MS-ADL with zero $\overline{0} = (0^{'}, 0)$ under point-wise operations. Consider $I = \{(0^{'}, 0), (0^{'}, a), (0^{'}, b), (0^{'}, c), (0^{'}, d), (a^{'}, 0), (a^{'}, a), (a^{'}, b), (a^{'}, c), (a^{'}, d)\}$ and $F = \{(0^{'}, e), (0^{'}, 1), (a^{'}, e), (a^{'}, 1)\}$. Clearly,$I$ is an ideals of L and$F$ is a filter of L. Now $\delta(F) = \{x \in L : x^\circ \in F\} = \{(0^{'}, 0), (0^{'}, a), (0^{'}, b), (0^{'}, c), (0^{'}, d), (a^{'}, 0), (a^{'}, a), (a^{'}, b), (a^{'}, c), (a^{'}, d)\} = I$. Therefore I is δ-ideal of L.

Lemma 3.6. *A proper δ-ideal of L contains no dense element.*

Proof. Let I be a proper δ-ideal L. Then by Lemma 3.3(4), there is a proper filter F of L such that $I = \delta(F)$. Now we need to show there is no dense element in I. Suppose not. Then there is a dense element $x \in \delta(F)$. This implies $0 = x^\circ \in F$. This shows that $F = L$. Which is a contradiction. Hence the result. □

Now, let us denote the set of all δ-Ideals of L by $I^\delta(L)$. Then, in the following Theorem, we prove that $I^\delta(L)$ forms a complete distributive lattice.

Theorem 3.7. *The set $I^\delta(L)$ forms a complete distributive lattice.*

Proof. For any two filters F, G of L, define two binary operations $\cap$ and $\sqcup$ as:

$$\delta(F) \cap \delta(G) = \delta(F \cap G) \text{ and}$$
$$\delta(F) \sqcup \delta(G) = \delta(F \vee G).$$

Now we prove $\delta(F) \sqcup \delta(G)$ is the supremum of $\delta(F), \delta(G) \in I^\delta(L)$. Since $F \subseteq F \vee G$ and $G \subseteq F \vee G$, by Lemma 3.3(5), we have $\delta(F), \delta(G) \subseteq \delta(F \vee G)$. This implies that $\delta(F \vee G)$ is an upper bound of $\delta(F), \delta(G)$. Let J be any δ-ideal containing $\delta(F)$ and $\delta(G)$. Then there exists a filter H of L such that $J = \delta(H)$ and $\delta(F) \subseteq \delta(H)$ and $\delta(G) \subseteq \delta(H)$. Now we prove that $\delta(F \vee G) \subseteq \delta(H)$. Let $x \in \delta(F \vee G)$. Then $x^\circ \in F \vee G$ and hence $x^\circ = f \wedge g$, for some $f \in F$ and $g \in G$. Thus by Lemma 3.3(3), we have $f^\circ \in \delta(F) \subseteq \delta(H)$ and $g^\circ \in \delta(G) \subseteq \delta(H)$.

$$\begin{aligned} &\Rightarrow \quad f^\circ \vee g^\circ \in \delta(H) \\ &\Rightarrow \quad x^{\circ\circ} = (f \wedge g)^\circ \in \delta(H) \\ &\Rightarrow \quad x^\circ \in H \\ &\Rightarrow \quad x \in \delta(H) \end{aligned}$$

Thus $\delta(F \vee G) \subseteq \delta(H)$. So $\delta(F \vee G)$ is the supremum of $\delta(F)$ and $\delta(G)$ in $I^\delta(L)$. Hence

$(I^\delta(L), \cap, \sqcup)$ is a lattice.
We now prove the distributivity.

Let $\delta(F), \delta(G), \delta(H)$. Then

$$\begin{aligned}\delta(F) \sqcup (\delta(G) \cup \delta(H)) &= \delta((F \vee G) \cap (F \vee H)) \\ &= \delta(F \vee G) \cap \delta(F \vee H) \\ &= (F \sqcup G) \cap (F \sqcup H)\end{aligned}$$

Thus $I^\delta(L)$ is a distributive lattice.
Next, we prove the completeness. Since $\{0\}$ and L are the least and greatest elements of $I^\delta(L)$. Let $\{\delta(F_i) : i \in I\}$ be a subfamily of $I^\delta(L)$. Then $\bigcap_{i\in I} \delta(F_i)$ is an ideal of L.

$$\begin{aligned} x \in \bigcap_{i\in I} \delta(F_i) \quad &\Leftrightarrow \quad x^\circ \in F_i \\ &\Leftrightarrow \quad x^\circ \in \bigcap_{i\in I} F_i \\ &\Leftrightarrow \quad x \in \delta(\bigcap_{i\in I} F_i)\end{aligned}$$

Thus $\bigcap_{i\in I} \delta(F_i) \in I^\delta(L)$. So $(I^\delta(L), \sqcup, \cap)$ is a complete distributive lattice. □

Definition 3.8. A δ-ideal I of L is called principal δ-ideal if there exists $x \in L$ such that $I = \delta([x))$.

Theorem 3.9. *For any $x \in L$, $(x^\circ]$ is a δ-ideal of L.*

Proof. Let $a \in (x^\circ]$. Then $a = x^\circ \wedge a$. This implies $a^\circ = x^{\circ\circ} \vee a^\circ$. This implies $a^\circ \wedge x = (x^{\circ\circ} \vee a^\circ) \wedge x = (x^{\circ\circ} \wedge x) \vee (a^\circ \wedge x) = x \vee (a^\circ \wedge x) = x$. Thus $a^\circ \in [x)$. So $a \in \delta([x))$. Hence $(x^\circ] \subseteq \delta([x))$.

Conversely, let $a \in \delta([x))$. Then $a^\circ \in [x)$ and so $a^\circ \vee x = a^\circ$.

$$\begin{aligned} &\Rightarrow \quad a^{\circ\circ} \wedge x^\circ = a^{\circ\circ} \\ &\Rightarrow \quad a^{\circ\circ} \wedge x^\circ \wedge a = a^{\circ\circ} \wedge a \\ &\Rightarrow \quad x^\circ \wedge a^{\circ\circ} \wedge a = a^{\circ\circ} \wedge a \\ &\Rightarrow \quad x^\circ \wedge a = a \\ &\Rightarrow \quad a \in (x^\circ]\end{aligned}$$

Hence $\delta([x)) \subseteq (x^\circ]$. Therefore $\delta([x)) = (x^\circ]$. □

Some properties of principal δ-ideals are given in the following:

Theorem 3.10. (1) *For all $a \in L$, $\delta([a)) = (a^\circ]$,*

(2) *For all $a \in L$, $\delta([a)) = \delta([a^{\circ\circ}))$,*

(3) *For all $d \in D$, $\delta([d)) = \{0\}$,*

(4) *For all $x \in F$, $\delta([x)) \subseteq \delta([F)$ for any filter F of L.*

Proof. (1) It is clear.

(2) Since $a^\circ = a^{\circ\circ\circ}$ and by (1), $\delta([a)) = (a^\circ] = (a^{\circ\circ\circ}] = \delta([a^{\circ\circ}))$.

(3) For every $d \in D$, we have $\delta([d)) = (d^\circ] = (0] = \{0\}$

(4) Let $x \in F$. Suppose that $y \in \delta([x))$. Then $y^\circ \in [x)$.

$$\begin{aligned} &\Rightarrow \quad y^\circ = y^\circ \vee x \in F \\ &\Rightarrow \quad y \in \delta(F)\end{aligned}$$

Hence for all $x \in F$, $\delta([x)) \subseteq \delta([F)$ for any filter F of L. □

Let us denote that the set of all principal δ-ideals of an MS-ADL L by $M^\circ(L) = \{\delta([x)) : x \in L\} = \{(x^\circ] : x \in L\}$. Then, in the following Theorem, it is observed that $M^\circ(L)$ is a de Morgan algebra.

Theorem 3.11. *$M^\circ(L)$ is a sublattice of the lattice $I^\delta(L)$ of all δ-Ideal of L and $M^\circ(L)$ can be made in to a de Morgan algebra. Moreover, the mapping $x \to (x^\circ]$ is a dual homomorphism of L into $M^\circ(L)$.*

Proof. Let $\delta([x)), \delta([y)) \in M^\circ(L)$ for some $x, y \in L$. Then we get $\delta([x)) \cap \delta([y)) = \delta([x \vee y)) \in M^\circ(L)$ and $\delta([x)) \vee \delta([y)) = \delta([x \vee y)) \in M^\circ(L)$. Also, $\{0\} = \delta([m)) \in M^\circ(L)$ and $L = \delta([0)) \in M^\circ(L)$. Hence $M^\circ(L)$ is a bounded sublattice of $I^\delta(L)$ and hence a distributive lattice. Now, define a unary operation on $M^\circ(L)$ by $\overline{\delta([x))} = \delta([x^\circ))$. We can easily verify that the unary operation "$-$" satisfies the de Morgan law. This indicates that $M^\circ(L)$ is de Morgan algebra. Also easily show that the mapping $x \to (x^\circ]$ is a dual homomorphism of L into $M^\circ(L)$. $\square$

Theorem 3.12. *For any ideal I of L, the following conditions are equivalent:*

(1) *I is a δ-Ideal,*

(2) *$I = \cup_{a\in I}\delta([a^\circ))$,*

(3) *for any x, y in L, $\delta([x^\circ)) = \delta([y^\circ))$ and $x \in I$ imply $y \in I$.*

Proof. (1) $\Rightarrow$ (2): Let I be a δ-ideal. Then $I = \delta(F)$ for some filter F of L. Let $x \in I$. Then $x \in \delta(F)$.

$$\begin{aligned} &\Rightarrow \quad x^{\circ\circ\circ} = x^\circ \in F \\ &\Rightarrow \quad x^{\circ\circ} \in \delta([x^\circ)) \\ &\Rightarrow \quad x \in \delta([x^\circ)) \subseteq \cup_{a\in I}\delta([a^\circ)) \end{aligned}$$

This implies $I \subseteq \cup_{a\in I}\delta([a^\circ))$. Clearly $\cup_{a\in I}\delta([a^\circ)) \subseteq I$. Hence $I = \cup_{a\in I}\delta([a^\circ))$.

(2) $\Rightarrow$ (3): Suppose condition (2) holds. Let for any x, y in L, $\delta([x^\circ)) = \delta([y^\circ))$ and $x \in I$. Then $\delta([x^\circ)) = \delta([y^\circ)) \subseteq I$.

$$\begin{aligned} &\Rightarrow \quad y^{\circ\circ} \in I \\ &\Rightarrow \quad y = y^{\circ\circ} \wedge y \in I \end{aligned}$$

(3) $\Rightarrow$ (1): Assume the condition (3). Let I be an ideal of L.
Consider a set

$$H = \{x \in L : x^\circ \in I\}.$$

Let $x, y \in H$. Then $x^\circ, y^\circ \in I$ and $(x \wedge y)^\circ \in I$. Thus $x \wedge y \in H$. Again, let $x \in H$ and $z \in L$. Then $x^\circ \in I$ and $x^\circ \wedge z^\circ \in I$. This implies $z \vee x \in H$. Thus H is a filter of L.
Now $(z \vee x)^\circ = (x \vee z)^\circ = x^\circ \wedge z^\circ \in I$. This implies $z \vee x \in H$ and H is a filter of L. Now we prove that $I = \delta(H)$. Let $x \in I$. Then we get $x \in I$ and $\delta([x^\circ)) = \delta([x^{\circ\circ\circ}))$. By condition (3), $x^{\circ\circ} \in I$.

$$\begin{aligned} &\Rightarrow \quad x^\circ \in H \\ &\Rightarrow \quad x \in \delta(H) \end{aligned}$$

This implies $I \subseteq \delta(H)$.

Conversely, let $x \in \delta(H)$. Then $x^\circ \in H$.

$$\begin{aligned} &\Rightarrow \quad x^{\circ\circ} \in I \\ &\Rightarrow \quad x = x^{\circ\circ} \wedge x \in I \end{aligned}$$

Thus $\delta(H) \subseteq I$. So $\delta(H) = I$. $\square$

4 δ-Ideals and Homomorphisms of MS-Almost Distributive lattice

In this section, some properties of the homomorphic images and the inverse images of δ-Ideals of MS-ADL are studied. By a homomorphism on an MS-ADL L, we mean a lattice homomorphism h satisfying $(f(x))^\circ = f(x^\circ)$ for all $x \in L$.

Theorem 4.1. *Let$f : L \to M$ be a homomorphism of an MS-ADL L onto an MS-ADL M. Then we have:*

(1) *For any δ-Ideal I of L, $f(I)$ is a δ-Ideal of M,*

(2) *For any $a \in L$, $f(\delta([a))) = \delta(f([a))$,*

(3) *For any δ-Ideal I of L, $f(I) = \cup_{i\in I}\delta([((f(i))^\circ)))$,*

(4) *For any filter F of L, $f(\delta(F)) = \delta(f(F))$.*

Proof. Let I be a δ-ideal of L. Then $I = \delta(F)$ for some filter F of L. Now,it is enough to show that $f(I) = \delta(f(F))$ for some filters of F.

$$\begin{aligned} y \in f(I) =& f(\delta(F)) \\ &\Rightarrow y = f(x) \;\; \textit{for some } x \in \delta(F) \\ &\Rightarrow x^\circ \in F \\ &\Rightarrow y^\circ = f(x)^\circ \in f(F) \\ &\Rightarrow y \in \delta(f(F)) \end{aligned}$$

This implies $f(I) \subseteq \delta(f(F))$.

Conversely, let $y \in \delta(f(F))$. Then $y^\circ = f(x)$ for some $x \in F$. This implies that $x^{\circ\circ} \in F$ and $x^\circ \in \delta(F)$. Thus $y^{\circ\circ} = f(x^\circ) \in f(\delta(F))$. Since $f(\delta(F))$ is an ideal of M and $y \in M$, we get that $y = y^{\circ\circ} \wedge y \in f(\delta(F))$. Thus $\delta(f(F)) \subseteq f(I)$. So $f(I) = \delta(f(F))$. $\square$

Theorem 4.2. *Let $f : L \to M$ be a homomorphism of an MS-ADL L into an MS-ADL M. Then we have:*

(1) *For any δ-ideal I of M, $f^{-1}(I)$ is a δ-ideal of L,*

(2) *$Kerf$ is a δ-ideal of L.*

Proof. (1) Since I is a δ-ideal of M, then $I = \delta(F)$ for some filter F of M. We prove that $f^{-1}(I) = \delta(f^{-1}(F))$, where $f^{-1}(I)$ is an ideal of L. Now,

$$\begin{aligned} x \in f^{-1}(I) &\Rightarrow f(x) \in I = \delta(F) \\ &\Rightarrow (f(x))^\circ = f(x^\circ) \in F \\ &\Rightarrow x^\circ \in f^{-1}(F) \\ &\Rightarrow x \in \delta(f^{-1}(F)) \end{aligned}$$

This implies $f^{-1}(I) \subseteq \delta(f^{-1}(F)$.

Conversely,

$$\begin{aligned} x \in \delta(f^{-1}(F)) &\Rightarrow x^\circ \in f^{-1}(F) \\ &\Rightarrow (f(x))^\circ \in F \\ &\Rightarrow f(x) \in \delta(F) = I \\ &\Rightarrow x \in f^{-1}(I) \\ &\Rightarrow \delta(f^{-1}(F)) \subseteq f^{-1}(I) \end{aligned}$$

Hence $f^{-1}(I)$ is a δ-ideal of L.

(2) Since f is a homomorphism, then $Kerf = \{x \in L : f(x) = 0\}$ is an ideal of L and $Cokerf = \{x \in L : f(x) = m\}$ is filter of L, for a maximal element of m. Easily we prove that $Kerf = \delta(Cokerf)$ and so $Kerf$ is δ-ideal of L. □

Lemma 4.3. *Let $f : L \to M$ be an onto homomorphism between MS-ADLs L and M. Then we have:*

(1) *$M^\circ(L)$ is homomorphic of $M^\circ(M)$,*

(2) *$I^\delta(L)$ is homomorphic of $I^\delta(M)$.*

Conflicts of Intersets

The author(s) declare(s) that there are no conflicts of interest regarding the publication of this paper.

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