Print ISSN 2710-0200 *Full text available online at* *Ethiop J Nat Comp Sci.*
Electronic ISSN 2710-0219 *http://journal.uog.edu.et/index.php/EJNCS* *Vol. 2 (2) 311-316*
***https://doi.org/**10.20372/ejncs/Ma.2022.03*

ORIGINAL RESEARCH

Fractions in Abstract R-vector spaces

Litegebe Wondie[1*], K.Venkateswarlu[2] , Zelalem Teshome[3] ,
[1]Department of Mathematics, College of Natural and Computational Sciences, University of Gondar, Gondar, Ethiopia
[2]Department of Basic Sciences and Humanities, College of Engineering Visakhapatnam, India
[3]Department of Mathematics Addis Ababa University, Addis Ababa, Ethiopia
*Corresponding author: E-mail: litgebihw2012@gmail.com

Received: 14 March 2020; Accepted: 24 November 2021; Published online: 15 August 2022

Abstract

In this paper, we introduce the notion of fractions in Abstract R-vector space, its norm, and obtaining certain properties. Also, we show that fraction of Abstract R-vector spaces is once again a vector space over fractions of the commutative regular ring with unity. Further, we introduce the notion of sub-vector spaces in a fraction of Abstract, R-vector spaces.

Keywords: Abstract R-vector spaces, Fractions in Abstract R-Vector Spaces, norm, sub-S^{-1} R-vector space.

Introduction

An Abstract R-Vector space is quite different from an ordinary R-module as it is a generalization of the Boolean vector space of Stroup's (1969) work. The concept of Abstract R-vector spaces has been studied by many authors in different ways. Rao (1966a; 1966b) has intensively studied it. Imitating the line of thought of Rao (1966a; 1966b) we introduce the notions of Fractions of Abstract R-vector spaces, its norm, its sub-vector spaces, and study their properties. This paper consists of four sections. In section one, we recall certain definitions and results concerning Abstract R-vector spaces. In section two, we introduce the notion of fractions of Abstract R-vector spaces and show that fractions of Abstract R-vector spaces are a vector space over fractions of the commutative regular ring. In section three, we establish the norm of a fraction of Abstract R-vector spaces and study its properties. Finally, in section four, we introduce the sub-vector space of a vector space over a fraction of commutative regular rings and establish that $S^{-1}V/S^{-1}U$ is isomorphic to S^{-1} (V/U) where U is a sub-Abstract R-vector space of an Abstract R-vector space V and S is a multiplicatively closed subset of a commutative regular ring R

1 Preliminaries

Here we collect certain definitions and results concerning fractions of commutative regular rings and Abstract R-vector spaces (vector space over regular rings) (Rao, 1966). Throughout this paper R stands for commutative regular rings with 1, B denotes the set of all idempotents of R, S stands for a non zero multiplicatively closed set. For further reference, the readers are advised to refer Rao (1966a) and Rao (1966b). Here after throughout the discussion of this paper "R-vector space" means "Abstract R-vector space".

Definition 1.1. *A non empty subset S of a commutative regular ring R with 1 is said to be multiplicatively closed if $1 \in S$ and $ab \in S$ for all $a, b \in S$.*

Theorem 1.2. *Let S be a multiplicatively closed subset of R. Define a relation $\sim$ on $R \times S$ by $(r_1, s_1) \sim (r_2, s_2) \Leftrightarrow u(s_2 r_1) = u(s_1 r_2)$ for some $u \in S$, $\forall r_1, r_2 \in R$ and $s_1, s_2 \in S$. Then $\sim$ is an equivalence relation.*

Remark 1.3. *The equivalence class containing $(r, s) \in R \times S$ is denoted by $\frac{r}{s}$. The set of all equivalence classes in $R \times S$ is denoted by $S^{-1}R = \{\frac{r}{s} : r \in R, s \in S\}$.*

Lemma 1.4. *Let S be a multiplicatively closed subset of R.Then:*
1. For $r, p \in R$ and $s, t \in S$, $\frac{r}{s} = \frac{p}{t} \Leftrightarrow u(tr) = u(sp)$ for some $u \in S$.
2. $\frac{r}{t} = \frac{rt}{st} = \frac{tr}{st} = \frac{tr}{ts}, \forall r \in R$ and $t, s \in S$
3. $\frac{rs_1}{s_1} = \frac{rs_2}{s_2}, \forall r \in R$ and $s_1, s_2 \in S$
4. $\frac{s_1}{s_1} = \frac{s_2}{s_2}, \forall s_1, s_2 \in S$
5. $\{\frac{0}{s}\} = \{\overline{0}\} \in S^{-1}R$ for $0 \in R$
6. $\frac{s}{s} = \frac{1}{1} = \overline{1} \in S^{-1}R$ for $1 \in R$

Theorem 1.5. *Let S be a multiplicatively closed subset of a commutative regular ring R. Define the binary operations $+$ and $\cdot$ on $S^{-1}R$ as $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{s_2 r_1 + s_1 r_2}{s_1 s_2}$ and $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2} \forall r_1, r_2 \in R$ and $s_1, s_2 \in S$. Then $S^{-1}R$ is a commutative regular ring.*

Remark 1.6. *The usual partial ordering, $<$, on B is defined as $a < b \Leftrightarrow ab = a$.*

Lemma 1.7. *Let R be a regular ring. For each $a \in R$,*
(i). $|a|a = a = a|a|$ (ii).$|a| = a \Leftrightarrow a \in B$.

Definition 1.8. *Let $V = (V, +)$ be an abelian group and $R = (R, +, .)$ be a commutative regular ring with unity element 1. Then V is said to be a Vector space over R (or simply R-vector space) if and only if there exists a mapping $:R \times V \to V$ (the image of any $(a, x) \in R \times V$ will be denoted by ax) such that for all $x, y \in V$ and $a, b \in R$, all the following properties hold:*

1. *$a^2(x + y) = ax + ay$*
2. *$a(bx) = (ab)x$ if $a^2 = a$*
3. *$1x = x$*
4. *$(a + b)x = ax + bx$ if $ab = 0$*
5. *$r(sx) = (rs)x$ if r and s are invertible elements of R*

Definition 1.9. *An R-vector space V is said to be normed if and only if there exists a mapping $||:V \to B$ satisfying the following properties.*
(1). $|x| = 0 \Leftrightarrow x = 0$ and (2). $|ax| = a|x|$ for all $x \in V, a \in B$

Corollary 1.10. *If V is normed R-vector space, then $|x|x = x$ for each $x \in V$.*

Lemma 1.11. *If V is a normed R-vector space, then $|x + y| < |x| + |y| - |x||y|$ for all $x, y \in V$.*

Definition 1.12. *Let W and U be R-vector spaces. Then the mapping*
$T : W \to U$ is a linear homomorphism if $T(ax + by) = aTx + bTy$ for all $a, b \in R$and $ab = 0$.

The set of linear homomorphism is denoted by $Hom(W, U)$.

Definition 1.13. *Let W and U be R-vector spaces. Then the mapping*
$T : W \to U$ is a strongly linear homomorphism if $T(ax + by) = aTx + bTy$ for all $a, b \in R$.

2 Construction of Fractions in R-Vector Spaces

In this section we construct fractions in R-vector spaces and study certain properties.

Theorem 2.1. *Let V be an R-Vector Space, $S = B$ be the set of all idempotents of R. Now define a relation $\sim$ on $V \times S$ as $(x, s) \sim (y, t) \Leftrightarrow u(tx) = u(sy)$ for some $u \in S$. Then $\sim$ is an equivalence relation.*

Proof. (i). For any $u \in S$, $u(sx) = u(sx). \Rightarrow (x, s) \sim (x, s)$.(ii). Let $(x_1, s_1) \sim (x_2, s_2). \Rightarrow u(s_2x_1) = u(s_1x_2)$ for some $u \in S. \Rightarrow u(s_1x_2) = u(s_2x_1) \Leftrightarrow (x_2, s_2) \sim (x_1, s_1)$.(iii). Let $(x_1, s_1) \sim (x_2, s_2)$ and $(x_2, s_2) \sim (x_3, s_3). \Rightarrow u(s_2x_1) = u(s_1x_2)$ and $v(s_3x_2) = v(s_2x_3)$ for some $u, v \in S$. Now, for $uvs_2 \in S, (uvs_2)(s_3x_1) = (uvs_2s_3)x_1 = (vs_3us_2)x_1 = vs_3(us_2x_1) = vs_3(us_1x_2) = us_1(vs_3x_2) = us_1(vs_2x_3) = (uvs_2)(s_1x_3). \Rightarrow (x_1, s_1) \sim (x_3, s_3)$. Hence $\sim$ is an equivalence relation. □

Remark 2.2. *The equivalence class containing $(x, s) \in V \times S$ is denoted by $\frac{x}{s}$. The set of all equivalence classes in $V \times S$ is denoted by $S^{-1}V = \{\frac{x}{s} : x \in V, s \in S\}$.*

Lemma 2.3. *Let V be an R-Vector Space, $S = B$ be the set of all idempotents of R. Then*
(i). $\frac{x_1}{s_1} = \frac{s_2x_1}{s_2s_1} = \frac{s_2x_1}{s_1s_2}$ for $s_1, s_2 \in S$ and $x_1 \in V$.
(ii). $\frac{s_1x_1}{s_1} = \frac{s_2x_1}{s_2} = x_1$ for $s_1, s_2 \in S$ and $x_1 \in V$.
(iii). $\frac{0}{s} = \frac{0}{t} = \overline{0}$ for any $t, s \in S$.
(iv). $\frac{x}{s} = \frac{0}{t} \Leftrightarrow ux = 0$ for some $u \in S$, $s, t \in S$ and $x \in V$.

Theorem 2.4. *Let V be an R-vector space and $S = B$ be the set of all idempotents of R. Define the binary operations addition and scalar multiplication on $S^{-1}V$ as follows: $\frac{x}{s} + \frac{y}{t} = \frac{tx+sy}{st}$ and $\frac{r}{s} \odot \frac{x}{t} = \frac{|r|x}{st}, x, y \in V, s, t \epsilon S, r \in R$. Then $S^{-1}V$ is an $S^{-1}R$-vector space.*

Proof. Since $\frac{0}{s} = \overline{0} \in S^{-1}V, S^{-1}V \neq \emptyset$.
To show " + " is well defined, let $\frac{x_1}{s_1}, \frac{x_2}{s_2}, \frac{x_3}{s_3}, \frac{x_4}{s_4} \in S^{-1}V$ such that $\frac{x_1}{s_1} = \frac{x_2}{s_2}, \frac{x_3}{s_3} = \frac{x_4}{s_4} \Leftrightarrow \exists u, v \in S$ such that $u(s_2x_1) = u(s_1x_2)$ and $v(s_4x_3) = v(s_3x_4)$. Now $(uv)[(s_2s_4)(s_3x_1 + s_1x_3)] = (vs_4s_3)u(s_2x_1) + u(s_2s_1)(vs_4x_3) = (vs_4s_3)u(s_1x_2) + u(s_2s_1)(vs_3x_4) = (uv)[(s_1s_3)(s_4x_2 + s_2x_4)]$.Thus " + " is well defined.
Now,let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R$ and $\frac{x}{t_1}, \frac{y}{t_2} \in S^{-1}V$.For some $uv \in S$ we have $(uv)[(s_2t_2)(|r_1|x)] = (u(s_2|r_1|))(v(t_2x)) = (u(s_1|r_2|))(v(t_1y)) = (uv)[(s_1t_1)(|r_2|y)]$.Thus " $\odot$ " is also well defined.
It is routine to verify that $(S^{-1}V, +)$ is an abelian group and with scalar multiplication $\odot$ it is routine to verify the axiom 1 through 5 of definition 1.7. □

3 Normed $S^{-1}R$-vector spaces

Here we introduce norm on fractions in R-vector spaces($S^{-1}R$-vector spaces) and obtain certain properties. We denote the set of all idempotents of $S^{-1}R$ by $B_{S^{-1}R}$.

Definition 3.1. *Let V be a normed Vector space over R and $S = B_{S^{-1}R}$ be a multiplicatively closed subset of R. A vector space $S^{-1}V$ over $S^{-1}R$ is said to be normed provided there exists a mapping $\|.\| : S^{-1}V \to B_{S^{-1}R}$ defined by $\|\frac{x}{s}\| = \frac{|x|}{|s|}$, satisfying:*
(1). $\|\frac{x}{s}\| = \frac{0}{s} = \overline{0} \Leftrightarrow \frac{x}{s} = \frac{0}{s} = \overline{0}$
(2). $\|\frac{r}{t}\frac{x}{s}\| = \frac{r}{t}\|\frac{x}{s}\|$, for all $\frac{r}{t} \in B_{S^{-1}R}$ and $\frac{x}{s} \in S^{-1}V$.

Corollary 3.2. *If $S^{-1}V$ is a normed $S^{-1}R$-vector space, then $\|\frac{x}{s}\|\frac{x}{s} = \frac{x}{s}$ for each $\frac{x}{s} \in S^{-1}V$.*

Proof. Using corollary 1.10,$\|\frac{x}{s}\|\frac{x}{s} = \frac{|x|}{|s|}\frac{x}{s} = \frac{|x|x}{ss} = \frac{x}{s}$. □

Lemma 3.3. *If $S^{-1}V$ is a normed vector space over $S^{-1}R$, then $\|\frac{x}{s} + \frac{y}{t}\| < \|\frac{x}{s}\| + \|\frac{y}{t}\| - \|\frac{x}{s}\|\|\frac{y}{t}\|$ for all $\frac{x}{s}, \frac{y}{t} \in S^{-1}V$.*

Proof. By definition 1.9 and lemma 1.11,$\|\frac{x}{s} + \frac{y}{t}\| = \|\frac{tx+sy}{st}\| = \frac{|tx+sy|}{|st|} < \frac{|tx|+|sy|-|tx||sy|}{st} = \frac{t|x|}{st} + \frac{s|y|}{st} - \frac{t|x|s|y|}{stst}$
$= \frac{|x|}{s} + \frac{|y|}{t} - \frac{|x|}{s}\frac{|y|}{t} = \|\frac{x}{s}\| + \|\frac{y}{t}\| - \|\frac{x}{s}\|\|\frac{y}{t}\|$. □

Definition 3.4. *If $S^{-1}V$ is a normed vector space over $S^{-1}R$, then*
(i). $[S^{-1}V] = \{\|\frac{x}{s}\| : \frac{x}{s} \in S^{-1}V\}$
(ii). $S^{-1}V_{\frac{r}{s}} = \{\frac{x}{t} : \|\frac{x}{t}\| < \frac{|r|}{s}\}$ for each $\frac{x}{t} \in S^{-1}V$ and $\frac{r}{s} \in S^{-1}R$.

Lemma 3.5. *If $S^{-1}V$ is a normed vector space over $S^{-1}R$, then $\|\frac{-x}{s}\| = \|\frac{x}{s}\|$ for each $\frac{x}{s} \epsilon S^{-1}V$.*

Proof. it is trivial □

Lemma 3.6. *If $S^{-1}V$ is a normed vector space over $S^{-1}R$ and $\|\frac{x}{s}\|\|\frac{y}{t}\| = \frac{0}{s} = \overline{0}$, then*
(i). $\|\frac{x}{s}\|\frac{y}{t} = \frac{0}{s} = \overline{0}$
(ii). $\|\frac{x}{s} + \frac{y}{t}\| = \|\frac{x}{s}\| + \|\frac{y}{t}\|$
(iii). $[S^{-1}V] = \{\|\frac{x}{s}\| : \frac{x}{s} \in S^{-1}V\}$ is an ideal of $B_{S^{-1}R}$ for all $\frac{x}{s}, \frac{y}{t} \in S^{-1}V$.

Proof. (i). Since $\|\|\frac{x}{s}\|\frac{y}{t}\| = \|\frac{x}{s}\|\|\frac{y}{t}\| = \frac{0}{s}, \|\frac{x}{s}\|\frac{y}{t} = \frac{0}{s}$.
(ii) By remark 1.6 and corollary 3.2,$(\|\frac{x}{s}\| + \|\frac{y}{t}\|)\|\frac{x}{s} + \frac{y}{t}\| = \|\frac{x}{s}\|\|\frac{x}{s} + \frac{y}{t}\| + \|\frac{y}{t}\|\|\frac{x}{s} + \frac{y}{t}\| = \|\|\frac{x}{s}\|\frac{x}{s} + \|\frac{x}{s}\|\frac{y}{t}\| + \|\|\frac{y}{t}\|\frac{x}{s} + \|\frac{y}{t}\|\frac{y}{t}\| = \|\frac{x}{s}\| + \|\frac{y}{t}\|$. On the other hand, by lemma 3.3, $\|\frac{x}{s} + \frac{y}{t}\| < \|\frac{x}{s}\| + \|\frac{y}{t}\|$.
(iii) Let $a, b \in [S^{-1}V]$ such that $\|\frac{x}{s}\| = a$ and $\|\frac{y}{t}\| = b$.$\|(1-b)\frac{x}{s}\|\|(1-a)\frac{y}{t}\| = (1-b)\|\frac{x}{s}\| + (1-a)\|\frac{y}{t}\| = (1-b)a(1-a)b = \overline{0}, \Rightarrow \|(1-b)\frac{x}{s} + (1-a)\frac{y}{t}\| = \|(1-b)\frac{x}{s}\| + \|(1-a)\frac{y}{t}\| = (1-b)\|\frac{x}{s}\| + (1-a)\|\frac{y}{t}\| = (1-b)a + (1-a)b = a - ab + b - ab = a - b \in [S^{-1}V]$. Let $a \in [S^{-1}V]$ such that $a = \|\frac{x}{t}\|, \frac{x}{t} \in S^{-1}V$ and $\frac{r}{u} \in B_{S^{-1}R}$. Now $\frac{r}{u}a = \frac{r}{u}\|\frac{x}{s}\| = \|\frac{r}{u} \odot \frac{x}{s}\| = \|\frac{|r|x}{us}\| \in [S^{-1}V]$. □

Theorem 3.7. *If $S^{-1}V$ is a normed vector space over $S^{-1}R$ and $\frac{r}{s} \in S^{-1}R$, then $S^{-1}V_{\frac{r}{s}}$ is a sub vector space of $S^{-1}V$ over $S^{-1}R$.*

Proof. Since $\frac{0}{s} \in (S^{-1}V)_{\frac{r}{s}}, (S^{-1}V)_{\frac{r}{s}}$ is non empty. Let $\frac{x}{s}, \frac{y}{t} \in (S^{-1}V)_{\frac{r}{s}}$ and $\frac{q}{u} \in S^{-1}R$. By lemma 3.3,3.6 and definition 3.4,$\|\frac{x}{s} - \frac{y}{t}\| < \|\frac{x}{s}\| + \|-\frac{y}{t}\| - \|\frac{x}{s}\|\|\frac{y}{t}\| = \|\frac{x}{s}\| + \|\frac{y}{t}\| - \|\frac{x}{s}\|\|\frac{y}{t}\| < |\frac{r}{s}| + |\frac{r}{s}| - |\frac{r}{s}||\frac{r}{s}| = |\frac{r}{s}|$. Again, $\|\frac{q}{u} \odot \frac{x}{s}\| = |\frac{q}{u}|\|\frac{x}{s}\| < \frac{|q|}{u}\frac{|r|}{s} < \frac{|r|}{s}$. Thus $\frac{x}{s} - \frac{y}{t}, \frac{q}{u} \odot \frac{x}{s} \in (S^{-1}V)_{\frac{r}{s}}$. □

4 Sub vector spaces in fractions of R-vector spaces

In this section we introduce the concept of sub vector space in fractions of R-vector space and study certain properties. Now we introduce the definition of sub R-vector space of an R-vector space V in the following

Definition 4.1. *Let V be an R-vector space. A non empty subset W of V is called a sub R-vector space of V if*
(i).For $x, y \in W$,$x - y \in W$.
(ii). For $a \in R$ and $x \in W$,$|a|x \in W$.

Remark 4.2. *If V is an R-vector space,W is a sub R-vector space of V then it is clear that W itself is an R-vector space.*

Lemma 4.3. *Let V be an R-vector space and W,U be sub R-vector spaces of V over R. Then $W \cap U$ is a sub R-vector space of V over R.*

Proof. Obvious □

Definition 4.4. *Let $S^{-1}V$ be a vector space over $S^{-1}R$. A non empty subset*

$S^{-1}W$ of $S^{-1}V$ is called a sub $S^{-1}R$-vector space of $S^{-1}V$ provided:
(i). $\frac{x}{s}-\frac{y}{t}\in S^{-1}W$ for $\frac{x}{s},\frac{y}{t}\in S^{-1}W$.
(ii). $\frac{|r|x}{st}\in S^{-1}W$ for $\frac{r}{s}\in S^{-1}R$ and $\frac{x}{t}\in S^{-1}W$.

Lemma 4.5. *Let W be a sub R-vector space of a vector space V. Then*
(i). $S^{-1}W$ is a sub $S^{-1}R$-vector space of $S^{-1}V$.
(ii). $s^{-1}x\in S^{-1}W \Leftrightarrow tx\in W$ for some $t\in S$

Proof. (i). Let $\frac{x}{s},\frac{y}{t}\in S^{-1}W$ and $\frac{r}{u}\in S^{-1}R$. clearly $\frac{|r|x}{us}, \frac{x}{s}-\frac{y}{t}=\frac{tx-sy}{st}\in S^{-1}W$. (ii).suppose $\frac{x}{s}\in S^{-1}W$. $\Rightarrow \frac{tx}{ts}=\frac{x}{s}\in S^{-1}W$ for some $t\in S$. $\Rightarrow tx\in W$,$ts\in S$ for some $t\in S$. Suppose $tx\in W$ for some $t\in S$. $\Rightarrow \frac{x}{s}=\frac{tx}{ts}\in S^{-1}W$ for some $t,s\in S$. □

Lemma 4.6. *Let V be an R-vector space and W, U be sub vector spaces of V over R. Then:*
(i). $S^{-1}(W\cap U)=S^{-1}W\cap S^{-1}U$.
(ii). $S^{-1}(W+U)=S^{-1}W+S^{-1}U$.

Proof. (i). it is clear from the definition. For (ii). let $\frac{w}{s}\in S^{-1}W, \frac{u}{s}\in S^{-1}U$. $\Rightarrow \frac{w}{s}\in S^{-1}(W+U)$ and $\frac{u}{s}\in S^{-1}(W+U)$. Since $S^{-1}(W+U)$ is a sub $S^{-1}R$-vector space of $S^{-1}V$,$\frac{w}{s}+\frac{u}{s}\in S^{-1}(W+U)$. Let $\alpha\in S^{-1}(W+U)$. $\Rightarrow \alpha=\frac{w+u}{s}=\frac{w}{s}+\frac{u}{s}\in S^{-1}W+S^{-1}U$. □

Definition 4.7. *Let $S^{-1}W$ and $S^{-1}U$ be an $S^{-1}R$-vector spaces. A one to one mapping $S^{-1}T$ from $S^{-1}W$ on to $S^{-1}U$ is called an isomorphism provided:*
(i). $(S^{-1}T)(\frac{x}{s}+\frac{y}{t})=(S^{-1}T)(\frac{x}{s})+(S^{-1}T)(\frac{y}{t})$ for $\frac{x}{s},\frac{y}{t}\in S^{-1}W$.
(ii).$(S^{-1}T)(\frac{a}{s}\odot\frac{x}{t})=\frac{a}{s}\odot(S^{-1}T)(\frac{x}{t})$ for $\frac{a}{s}\in S^{-1}R$ and $\frac{x}{t}\in S^{-1}W$.

Theorem 4.8. *Let $S^{-1}W$ and $S^{-1}U$ be an $S^{-1}R$-vector spaces. A mapping $S^{-1}T: S^{-1}W\to S^{-1}U, (S^{-1}T)(\frac{x}{t})=\frac{Tx}{t}$, is an element of $Hom(S^{-1}W,S^{-1}U)\Leftrightarrow (S^{-1}T)(\frac{a}{s}\odot\frac{x}{t})=\frac{a}{s}\odot(S^{-1}T)(\frac{x}{t})$ for $\frac{a}{s}\in S^{-1}R$ and $\frac{x}{t}\in S^{-1}W$.*

Proof. Suppose $S^{-1}T\in Hom(S^{-1}W,S^{-1}U)$.If $\frac{x}{t}\in S^{-1}W$ and $\frac{a}{s}\in S^{-1}R$, then $(S^{-1}T)(\frac{a}{s}\odot\frac{x}{t}+\frac{0}{s}\odot\frac{x}{t})=\frac{a}{s}\odot(S^{-1}T)(\frac{x}{t})+\frac{0}{s}\odot(S^{-1}T)(\frac{x}{t})=\frac{a}{s}\odot(S^{-1}T)(\frac{x}{t})$. Suppose $(S^{-1}T)(\frac{a}{s}\odot\frac{x}{t})=\frac{a}{s}\odot(S^{-1}T)(\frac{x}{t})$. Let $ab=0$ and $(S^{-1}T)(\frac{a}{s_1}\odot\frac{x}{t_1}+\frac{b}{s_2}\odot\frac{y}{t_2}=(S^{-1}T)(\frac{|a|x}{s_1t_1}+\frac{|b|y}{s_2t_2})=\frac{T(|a|x)}{s_1t_1}+\frac{T(|b|y)}{s_2t_2}=\frac{a}{s_1}\odot(S^{-1}T)(\frac{x}{t_1})+\frac{b}{s_2}\odot(S^{-1}T)(\frac{y}{t_2})$. □

Theorem 4.9. *Let W and U be R-vector spaces and $T:W\to U$ be a strongly linear homomorphism.Then $S^{-1}T: S^{-1}W\to S^{-1}U, \frac{w}{s}\longmapsto\frac{T(w)}{s}, w\in W, s\in S$ is also a strongly linear homomorphism.*

Proof. Let $\frac{x}{t_1},\frac{y}{t_2}\in S^{-1}W$ and $\frac{a}{s_1},\frac{b}{s_2}\in S^{-1}R$.
Now
$(S^{-1}T)(\frac{a}{s_1}\odot\frac{x}{t_1}+\frac{b}{s_2}\odot\frac{y}{t_2})=(S^{-1}T)(\frac{|a|x}{t_1s_1}+\frac{|b|y}{t_2s_2})=(S^{-1}T)(\frac{(s_2t_2)(|a|x)+(s_1t_1)(|b|y)}{(s_1t_1)(s_2t_2)})=\frac{T[(s_2t_2)(|a|x)+(s_1t_1)(|b|y)]}{(s_1t_1)(s_2t_2)}=\frac{(s_2t_2)T(|a|x)+(s_1t_1)T(|b|y)}{(s_1t_1)(s_2t_2)}=\frac{|a|Tx}{s_1t_1}+\frac{|b|Ty}{s_2t_2}=\frac{a}{s_1}\odot(S^{-1}T)(\frac{x}{t_1})+\frac{b}{s_2}\odot(S^{-1}T)(\frac{y}{t_2})$. □

Corollary 4.10. *Let $S^{-1}W$ and $S^{-1}U$ be an $S^{-1}R$-vector spaces. If $S^{-1}T: S^{-1}W\to S^{-1}U,\frac{w}{s}\longmapsto\frac{T(w)}{s}$ is a strongly linear homomorphism, then kernel of $S^{-1}T$ is sub $S^{-1}R$-vector space of $S^{-1}W$.*

Proof. Let kernel of $S^{-1}T=\{\frac{x}{s}\in S^{-1}W:(S^{-1}T)(\frac{x}{s})=\frac{0}{s}\}$. Let $\frac{x}{s},\frac{y}{t}\in ker(S^{-1}T)$. Now, $(S^{-1}T)(\frac{x}{s}-\frac{y}{t})=(S^{-1}T)(\frac{tx-sy}{st})=\frac{T(tx-sy)}{st}=\frac{tTx-sTy}{st}=\frac{Tx}{s}-\frac{Ty}{t}=\bar{0}$. For $\frac{a}{s}\in S^{-1}R$ and $\frac{x}{t}\in ker(S^{-1}T)$, $(S^{-1}T)(\frac{a}{s}\odot\frac{x}{t})=(S^{-1}T)(\frac{|a|x}{st})=\frac{T(|a|x)}{st}=\frac{|a|Tx}{st}=\frac{a}{s}\odot\frac{Tx}{t}=\frac{a}{s}\odot(S^{-1}T)(\frac{x}{t})=\bar{0}$. Hence kernel of $S^{-1}T$ is sub $S^{-1}R$-vector space of $S^{-1}W$. □

Finally, we prove the following

Theorem 4.11. *Let V be an R-vector space and U be a sub R-vector space of V. Then $S^{-1}(V/U) \cong S^{-1}V/S^{-1}U$.*

Proof. Let $T : S^{-1}V/S^{-1}U \to S^{-1}(V/U)$ is a map defined by $T(\frac{x}{s} + S^{-1}U) = \frac{x+u}{s}$.
Let $T(\frac{x}{s} + S^{-1}U) = T(\frac{y}{s} + S^{-1}U)$ for $x, y \in V$ and $s \in S \Rightarrow \frac{x+u}{s} = \frac{y+u}{s} \Leftrightarrow \exists v \in S$ such that $v(s(x+u)) = v(s(y+u)) \Rightarrow \beta x + U = \beta y + U \Rightarrow \beta(x-y) \in U \Rightarrow \frac{\beta s(x-y)}{ss} \in S^{-1}U \Rightarrow \beta(\frac{x}{s} - \frac{y}{s}) \in S^{-1}U \Rightarrow \frac{x}{s} + S^{-1}U = \frac{y}{s} + S^{-1}U \Rightarrow T$ is one to one. Clearly T is on to.
Let $\frac{x}{s} + S^{-1}U, \frac{y}{t} + S^{-1}U \in S^{-1}V/S^{-1}U$. Now, $T(\frac{x}{s} + S^{-1}U + \frac{y}{t} + S^{-1}U) = T(\frac{x}{s} + \frac{y}{t} + S^{-1}U) = T(\frac{tx+sy}{st} + S^{-1}U) = \frac{(tx+sy)+u}{st} = \frac{t(x+u)}{ts} + \frac{s(y+u)}{st} = T(\frac{x}{s} + S^{-1}U) + T(\frac{y}{t} + S^{-1}U)$.
Let $\frac{a}{s} \in S^{-1}R$ and $\frac{x}{t} + S^{-1}U \in S^{-1}V/S^{-1}U$. $T(\frac{a}{s} \odot (\frac{x}{t} + S^{-1}U)) = T(\frac{|a|x}{st} + S^{-1}U) = \frac{|a|x+u}{st} = \frac{|a|(x+u)}{st} = \frac{a}{s} \odot (\frac{x+u}{t}) = \frac{a}{s} \odot T(\frac{x}{t} + S^{-1}U)$. □

References

[1] Stroup,T.O. (1969). *On the theory of Boolean Vector Spaces*, Doctoral Thesis, University of Missiouri, Columbia.

[2] Subrahmanyam, N.V. (1964). *Boolean Vector Spaces I*, Math. Zeit. 83:422-433.

[3] Rao,N.R.G.(1966a). *Vector spaces over regular rings*, Math. annalen, 167:280-291.

[4] Rao, N.R.G. (1966b). *Regular rings and Vector spaces over regular rings*, Doctoral Thesis, Andhra University,Waltair, India.

[5] Al-Taha,S.A (2013). *The Multiplicative closed set* , Pure Mathematical Sciences, HIKARI Ltd, Al Zaytoonah University,Amman Jordan, 115 - 120.